INTERPOLATION AND EXTREMAL THEOREMS: THE HAMILTONIAN NUMBER OF CUBIC GRAPHS

A THESIS

BY

SERMSRI THAITHAE

Presented in Partial Fulfillment of the Requirements for the

Doctor of Philosophy in Mathematics

at Srinakharinwirot University

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การปรากฏค่าทุกค่าและค่าสุดขีด :
จำนวนแฮมิลโทเนียนของกราฟลูกบาศก์

บทคัดย่อ

ของ

เสริมศรี ไทยแท้

เสนอต่อบัณฑิตวิทยาลัย มหาวิทยาลัยศรีนครินทรวิโรฒ เพื่อเป็นส่วนหนึ่งของการศึกษา
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ทางเดินแฮมิลโทเนียนในกราฟเชื่อมโยง G ที่มีอันดับ n คือ ทางเดินปิดแผ่ทั่วกราฟที่มีความยาวสั้นที่สุดในกราฟ G จำนวนแฮมิลโทเนียนของกราฟเชื่อมโยง G คือ ความยาวของทางเดินแฮมิลโทเนียนในกราฟเชื่อมโยง G และแทนด้วย h ดังนั้น h อาจจะใช้เป็นการวัดความห่างของการเป็นแฮมิลโทเนียนของกราฟ ซึ่งเป็นที่รู้กันโดยทั่วไปว่าปัญหาการหาว่ากราฟที่กำหนดให้เป็นแฮมิลโทเนียนหรือไม่เป็นปัญหาเอ็นพีบริบูรณ์ (NP-complete)

ให้ $\mathcal{J}$ เป็นคลาสของกราฟ เราศึกษาปัญหาของการหาเรนจ์ของจำนวนแฮมิลโทเนียน

$$h(\mathcal{J}) = \{h(G) : G \in \mathcal{J}\}$$

เราได้หาเรนจ์ของจำนวนแฮมิลโทเนียนทั้งหมดแล้วเมื่อ $\mathcal{J} = \mathcal{CR}(3^n)$ โดยที่ $\mathcal{CR}(3^n)$ เป็นคลาสของกราฟลูกบาศก์เชื่อมโยงทั้งหมดที่มีอันดับ n และ $\mathcal{J} = \mathcal{G}(n; \kappa = k)$ โดยที่ $\mathcal{G}(n; \kappa = k)$ เป็นคลาสของกราฟทั้งหมดที่มีอันดับ n กับค่าความเชื่อมโยง k ในส่วนสุดท้ายเราได้กำหนดปัญหาปลายเปิดสำหรับผู้สนใจที่จะทำการศึกษาทางด้านนี้

INTERPOLATION AND EXTREMAL THEOREMS: THE HAMILTONIAN NUMBER OF CUBIC GRAPHS

AN ABSTRACT

BY

SERMSRI THAITHAE

Presented in Partial Fulfillment of the Requirements for the

Doctor of Philosophy in Mathematics

at Srinakharinwirot University

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Sermsri Thaithae. (2009). *Interpolation and Extremal Theorems : The Hamiltonian Number of Cubic Graphs.* Dissertation, Ph.D. (Mathematics). Bangkok: Graduate School, Srinakharinwirot University. Advisor Committee: Prof. Dr. Narong Punnim, Prof. Dr. Dinesh G. Sarvate, Asst. Prof. Dr. Varaporn Saenpholphat.

A Hamiltonian walk in a connected graph G of order n is a closed spanning walk of minimum length in G. The Hamiltonian number $h(G)$ of a connected graph G is the length of a Hamiltonian walk in G. Thus h may be considered as a measure of how far a given graph is from being Hamiltonian. It is well known that the problem of determining whether a given graph is Hamiltonian or not is NP-complete

Let $\mathcal{J}$ be a class of graphs. We consider the problem of determining the range of Hamiltonian numbers

$$h(\mathcal{J}) = \{h(G) : G \in \mathcal{J}\}.$$

We completely solve the problem when $\mathcal{J} = \mathcal{CR}(3^n)$ where $\mathcal{CR}(3^n)$ is the class of all connected cubic graphs of order n and $\mathcal{J} = \mathcal{G}(n; \kappa = k)$ where $\mathcal{G}(n; \kappa = k)$ is the class of all graphs of order n with connectivity k. Open problems in this direction are also provided for future work.

The dissertation titled

"Interpolation and Extremal Theorems :

The Hamiltonian Number of Cubic Graphs"

by

Sermsri Thaithae

has been approved by the Graduate School as partial fulfillment of the requirements for the Doctor of Philosophy in Mathematics of Srinakharinwirot University.

.... Dean of Graduate School

(Assoc. Prof. Dr.Somchai Santiwatanakul)

June 4, 2009

Dissertation Committee

.... Major-advisor

(Prof. Dr.Narong Punnim)

.... Co-advisor

(Prof. Dr.Dinesh G. Sarvate)

.... Co-advisor

(Asst. Prof. Dr.Varaporn Saenpholphat)

Oral Defense Committee

.... Chair

(Dr.Sayun Sotaro)

.... Committee

(Prof. Dr.Narong Punnim)

.... Committee

(Prof. Dr.Dinesh G. Sarvate)

.... Committee

(Asst. Prof. Dr.Varaporn Saenpholphat)

.... Committee

(Asst. Prof. Dr.Chariya Uiyyasathian)

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Sermsri Thaithae

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CHAPTER 1

INTRODUCTION

1.1 Prologue

In the mathematical field of graph theory, a *Hamiltonian path* of graph is a path in a simple graph which contains every vertex of graph exactly once. A *Hamiltonian cycle* (or Hamiltonian circuit) of graph is a cycle which contains every vertex of graph exactly once and also returns to the starting vertex.

Hamiltonian paths and cycles are named after William Rowan Hamilton who invented the Icosian Game, now also known as Hamilton's puzzle, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron. Hamilton solved this problem using the Icosian Calculus, an algebraic structure based on roots of unity with many similarities to the quaternions (also invented by Hamilton). Unfortunately, this solution does not generalize to arbitrary graphs.

In general, the problem of determining whether such paths and cycles exist in graphs is the Hamiltonian path problem which is NP-complete [11]. The only known way to determine whether a given general graph has a Hamiltonian path or Hamiltonian cycle is to undertake an exhaustive search. Rubin [17] in 1974 describes an efficient search procedure that can find some or all Hamilton paths and circuits in a graph using deductions that greatly reduce backtracking and guesswork.

Questions relating to paths and cycles in graphs and digraphs have been extensively studied a variety of perspectives. One of the most popular questions in undirected graphs relates to determining the class of graphs which contain a Hamiltonian path or cycle. This basic question has been generalized and specialized in many ways which have led to obtain a number of deep and interesting results (for survey of such results, see [13]).

Our particular interest is to study the Hamiltonian number in some classes of connected graphs. The concept of Hamiltonian number of a connected graph was introduced by Goodman and Hedetniemi [12] in 1973. The Hamiltonian number of a connected graph G, denoted by $h(G)$, is the minimum length among all closed spanning walks in G. Therefore, if G is a graph of order n, then G is Hamiltonian if and only if $h(G) = n$. In general, if G is a connected graph of order n and not Hamiltonian, then a Hamiltonian walk might pass through some vertices and traverse some edges, more than once. In this case $h(G) \geq n + 1$. Thus the function h may be considered as a measure of how far a given graph is from being Hamiltonian. An upper bound for $h(G)$ of a connected graph G of order n was obtained also in [12], that is $h(G) \leq 2n - 2$ and $h(G) = 2n - 2$ if and only if G is a tree. Moreover, it was proved that for a given integer p where $n \leq p \leq 2n - 2$, there exists a connected graph G_p of order n and $h(G_p) = p$. This leads to the problem of determining of $h(\mathcal{J})$ seems to be more interesting for several classes of graphs, e.g., the class of connected r-regular graphs of order n, the class of connected graphs of order n and size m, the class of connected graphs of order n and diameter d, the class of connected graphs of order n, and connectivity k, etc. More precisely, let $\mathcal{CG}(n)$ be the class of connected graphs of order n and $\mathcal{J} \subseteq \mathcal{CG}(n)$. By putting $h(\mathcal{J}) := \{h(G) : G \in \mathcal{J}\}$, we call it the range of Hamiltonian numbers in $\mathcal{J}$. Evidently, if $n \geq 3$, then $h(\mathcal{CG}(n)) = \{x \in \mathbb{Z} : n \leq x \leq 2n - 2\}$. Since a connected 2-regular graph of order n is Hamiltonian, it is reasonable to obtain $h(\mathcal{CR}(3^n))$, where $\mathcal{CR}(3^n)$ is the class of connected cubic graphs of order n. In this dissertation, we focus on the determination of $h(\mathcal{CR}(3^n))$ and some other related subclasses.

1.2 Background

We present, in this section, the basic notation and terminology. For most part, our graph theoretic notation and terminology can be found in the textbooks of Chartrand and Lesniak [5] and Chartrand and Zhang [6]. In particular, *a graph* G consists of a set of vertices $V(G)$, a set of edges $E(G)$, and an incidence relation which associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G called its ends. If e is an edge of G with ends u and v, then e is said to join u and v. An edge with identical ends is called a *loop*. We can represent a graph by a diagram in which the vertices are points and edges are line segments. Thus, in general, there is no unique way of drawing a graph. Two vertices which are joined by an edge are said to be *adjacent*. If more than one edge joins the same pair of vertices, we say the graph has *multiple edges*. A graph is *simple* if it has no loops or multiple edges. We limit our discussion to graphs that are simple and finite.

Since we deal only with finite graphs, we set up the following notation and terminology for a typical graph G. For a graph G, we may write $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Denoting by $|S|$ the cardinality of a set S we define $n(G) = |V(G)|$ the *order* of G,or simply n, and $m(G) = |E(G)|$ the *size* of G or simply m. If an edge e corresponds to the vertex pair $\{u, v\}$, to simplify writing, we will write $e = uv$ and we say that the edge e *joins* the vertices u and v. For each $v \in V(G)$, a vertex $u \in V(G)$ is called a *neighbor* of v if $vu \in E(G)$. The *neighborhood* of $v \in V(G)$ in G, denoted by $N_G(v)$, is defined by

$$N_G(v) = \{x \in V(G) : xv \in E(G)\}.$$

If $S \subseteq V$, we denote $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_S(v) = N_G(v) \cap S$.

The idea of *sameness* or *identical* of graphs is formalized by the following definition.

Two graphs G and H are equal if $V(G) = V(H)$ and $E(G) = E(H)$. Two graphs G_1 and G_2 are *isomorphic* if there exists a one-to-one correspondence ϕ from $V(G_1)$ to $V(G_2)$ such that $u_1v_1 \in E(G_1)$ if and only if $\phi(u_1)\phi(v_1) \in E(G_2)$. In this case, ϕ is called an *isomorphism* from G_1 to G_2. Thus, if G_1 and G_2 are isomorphic graphs, then we say that G_1 is *isomorphic to* G_2 and we write $G_1 \cong G_2$. If two graphs G and H are not isomorphic, then they are called *nonisomorphic graphs* and we write $G \ncong H$.

It is clear that the relation " $\cong$ " is an equivalence relation on the set of all simple graphs. Consequently, it partitions the set of all graphs into equivalence classes each of which consists of all graphs having the same *algebraic properties*. The meaning of what we call algebraic property can be defined in the following definition.

Let $\mathcal{G}$ be the class of simple graphs and let X be any set. A function $f : \mathcal{G} \to X$ is called an *invariant* if $G \cong H$ then $f(G) = f(H)$. When $X = \mathbb{Z}$, a graph invariant is called a *graph parameter.*

A graph H is called a *subgraph* of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We also say that G *contains* H as a subgraph. If $H \subseteq G$ and either $V(H)$ is a *proper subset* of $V(G)$ or $E(G)$ is a proper subset of $E(G)$, then H is a proper subgraph of G. If a subgraph of a graph G has the same vertex set as G, then it is a *spanning subgraph* of G.

A subgraph F of a graph G is called an *induced subgraph* of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well. If S is a nonempty set of vertices of a graph G, then the *subgraph of G induced by S* is the induced subgraph with vertex set S. This induced subgraph is denoted by $\langle S \rangle$. To emphasize that this is an induced subgraph of G, we sometimes denote this subgraph by $\langle S \rangle_G$ or by $G[S]$.

Any proper subgraph of a graph G can be obtained by removing vertices and edges from G. For an edge e of G, we write $G - e$ for the spanning subgraph of G whose edge set consists of all edges of G except e. More generally, if X is a set of edges of G, then $G - X$ is the spanning subgraph of G with $E(G - X) = E(G) - X$. If $X = \{e_1, e_2, ..., e_k\}$, then we also write $G - X$ as $G - e_1 - e_2 - \cdots - e_k$.

For a vertex v of a nontrivial graph G, the subgraph $G - v$ consists of all vertices of G except v and all edges of G except those incident with v. For a proper subset U of $V(G)$, the subgraph $G - U$ has vertex set $V(G) - U$ and its edge set consists of all edges of G joining two vertices in $V(G) - U$. Necessarily, $G - U$ is an induced subgraph of G; indeed, $G - U = \langle V(G) - U \rangle$.

If u and v are nonadjacent vertices of a graph G, then $e = uv \notin E(G)$. By $G + e$, we mean the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. Thus G is a spanning subgraph of $G + e$.

A graph G is *complete* if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. Therefore, K_n has the maximum possible size for a graph with n veertices. Since every two distinct vertices of K_n are joined by an edge, the number of pairs of vertices in K_n is $\binom{n}{2}$ and so the size of

K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$.

The *complement* $\overline{G}$ of a graph G is the graph whose vertex set is $V(G)$ and such that for each pair u, v of vertices of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. Observe that if G is a graph of order n and size m, then $\overline{G}$ is a graph of order n and size $\binom{n}{2} - m$. The graph $\overline{K_n}$ then has n vertices and no edges; it is called the *empty graph* of order n. Therefore, empty graphs have empty edge sets.

A graph G is a *bipartite graph* if $V(G)$ can be partitioned into two subsets U and W, called *partite sets*, such that every edge of G joins a vertex of U and a vertex of W. We call G a *complete bipartite graph* if every vertex of U is adjacent to every vertex of W. A complete bipartite graph with $|U| = s$ and $|W| = t$ is denoted by $K_{s,t}$ or $K_{t,s}$. If either $s = 1$ or $t = 1$, then $K_{s,t}$ is *star*.

Let G_1 and G_2 be two disjoint graphs. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of $k(\geq 2)$ disjoint copies of a graph H, then we write $G = kH$. Let G and H be any two graphs, not necessarily disjoint. The *join* $G + H$ consists of $G \cup H$ and all edges joining a vertex of G and a vertex of H.

For two (not necessarily vertex-disjoint) graphs G and H, the *Cartesian product* $G \times H$ has vertex set $V(G \times H) = V(G) \times V(H)$, that is, every vertex of $G \times H$ is an ordered pair (u, v), where $u \in V(G)$ and $v \in V(H)$. Two distinct vertices (u, v) and (x, y) are adjacent in $G \times H$ if either $(1) u = x$ and $vy \in E(H)$ or $(2) v = y$ and $ux \in E(G)$. Notice that $K_2 \times K_2$ is the 4-cycle. The graph $C_4 \times K_2$ is often denoted by Q_3 and is called the 3-*cube*. More generally, we define Q_1 to be K_2 and for $n \geq 2$, define Q_n to be $Q_{n-1} \times K_2$. The graphs Q_n are then called *n-cubes* or *hypercubes*.

We now define a number of concept arising from the adjacency and incidence relations in a graph, leading to the concept of a connected graph.

A $u - v$ *walk* W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent, that is, we can express W as

$$W : u = v_0, v_1, ..., v_k = v,$$

where $k \geq 0$ and v_i and v_{i+1} are adjacent for $i = 0, 1, 2, ..., k-1$. Each vertex $v_i (0 \leq i \leq k)$ and each edge $v_i v_{i+1} (0 \leq i \leq k-1)$ is said to lie on or belong to W. If $u = v$ then the walk W is *closed*; while if $u \neq v$, then W is *open*. The number of edges encountered in a walk(including multiple occurrences of an edge) is called the *length* of the walk.

A $u - v$ *trail* in a graph G to be a $u - v$ walk in which no edge is traversed more than once. A $u - v$ walk in a graph in which no vertices are repeated is a $u - v$ *path*. A *circuit* in a graph G is a closed trail of length 3 or more. A circuit that repeats no vertex, except for the first and last, is a *cycle*. A path of order n is called an n-path and is denoted by P_n. A cycle of order n is called an n-cycle and is denoted by C_n.

If G contains a $u - v$ path, then u and v are said to be *connected* and u *is connected to* v (and v is connected to u). By convention, a vertex is connected to itself. A graph G is *connected* if every two vertices of G are connected, that is, if G contains a $u - v$ path for every pair u, v of distinct vertices of G. A graph G that is not connected is called *disconnected*. A connected subgraph of G that is not a proper subgraph of any other connected subgraph of G is a *component* of G. A graph G is then connected if and only if it has exactly one component. An edge $e = uv$ of a connected graph G is called a *bridge* of G if $G - e$ is disconnected.

A graph G is called *acyclic* if it has no cycles. A *tree* is an acyclic connected graph. A spanning subgraph H of a connected graph G such that H is a tree is called a *spanning tree* of G.

Let G be a connected graph of order n, and let u and v be two vertices of G. The *distance* between u and v is the smallest length of any $u - v$ path in G and is denoted by $d_G(u, v)$ or simply $d(u, v)$. The greatest distance between any two vertices of a connected graph G is called the *diameter* of G and is denoted by $\mathrm{diam}(G)$.

The *degree of a vertex* v of a graph G is the number of edges of G which are incident with v. In symbol

$$\deg(v) = |\{e \in E : e = uv \text{ for some } u \in V\}|.$$

The *minimum degree* and the *maximum degree* of a graph G are usually denoted by the special symbols $\delta(G)$ and $\Delta(G)$ respectively. So if G is a graph of order n and V is any vertex of G, then

$$0 \leq \delta(G) \leq \deg(v) \leq \Delta(G) \leq n-1.$$

A simple graph is said to be *r-regular* if all of its vertices have degree r. A 3-regular graph is usually called a *cubic graph.* A vertex with degree zero is called an *isolated vertex.* An r-regular spanning subgraph of a simple graph G is called an r-factor of G.

A vertex of odd degree is called an *odd vertex* and a vertex of even degree is called an *even vertex.* The following theorem is known as *The First Theorem of Graph Theory.*

Theorem 1.2.1 *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg(v) = 2m.$$

□

Corollary 1.2.2 *Every graph has an even number of odd vertices.* □

A vertex v in a connected graph G is a *cut-vertex* of G if $G-v$ is disconnected. A nontrival connected graph with no cut-vertices is called a *nonseparable graph.* A nonseparable subgraph of a graph G that is not a proper subgraph of any other nonseparable subgraph in G is called a *block.* Each block of G is an induced subgraph of G.

A *vertex-cut* in a graph G is a set U of vertices of G such that $G-U$ is disconnected. For a graph G that is not complete, the *vertex-connectivity* (or simply the *connectivity*) $\kappa(G)$ of G is defined as the cardinality of a minimum vertex-cut of G; if $G \cong K_n$ for some positive integer n, then $\kappa(G)$ is defined to be $n-1$. Therefore, for every graph G of order n,

$$0 \leq \kappa(G) \leq n - 1.$$

For a nonnegative integer k, a graph G is said to be *k-connected* if $\kappa(G) \geq k$. Therefore, a k-connected graph is also l-connected for every integer l with $0 \leq l \leq k$.

A set of vertices in a graph is *independent* if no two vertices in the set are adjacent. The *vertex independence number* (or the *independence number*) $\beta(G)$ of a graph G is the maximum cardinality of an independent set of vertices in G.

1.3 Hamiltonian Graphs and Numbers

A cycle in a graph G that contains every vertex of G is called a *Hamiltonian cycle* of G. A *Hamiltonian graph* is a graph that contains a Hamiltonian cycle. Certainly the graph C_n ($n \geq 3$) is Hamiltonian. Also, for $n \geq 3$, the complete graph K_n is a Hamiltonian graph. A path in a graph G that contains every vertex of G is called a *Hamiltonian path* in G. If a graph contains a Hamiltonian cycle, then it contains a Hamiltonian path. The graph $G \cong K_{3,3}$ of Figure 1 is a Hamiltonian graph.

G :

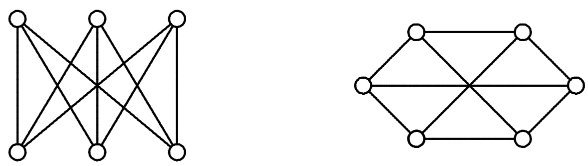


Figure 1 The Hamiltonian graph $K_{3,3}$.

The graph G of Figure 2 is not a Hamiltonian graph.

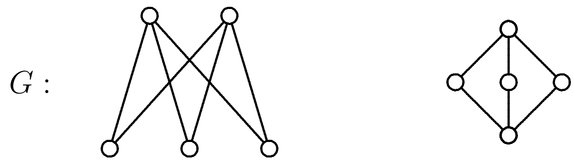


Figure 2 A non-Hamiltonian graph.

One of the most famous non-Hamiltonian graphs is the Petersen graph (shown in Figure 3). The Petersen graph is a cubic graph of order 10.

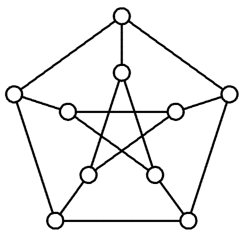


Figure 3 The Petersen graph.

The following result of Dirac [10] gives a sufficient condition graph to be Hamiltonian.

Theorem 1.3.1 *If G is a simple graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.* □

The concept of a Hamiltonian walk was introduced by Goodman and Hedetniemi [12] in 1973. A *Hamiltonian walk* in a connected graph G is a closed spanning walk of minimum length in G. The Hamiltonian number of a connected graph G, denoted by $h(G)$, is the length of a Hamiltonian walk in G. Therefore, for a connected graph G of order $n \geq 3$, it follows that $h(G) = n$ if and only if G is Hamiltonian. So, if G is non-Hamiltonian graph, then $h(G) \geq n + 1$. The Petersen graph is a non-Hamiltonian graph of order 10 with $h(G) = 11$.

CHAPTER 2

REVIEW OF THE LITERATURE

The purpose of this chapter is to review some relevant works on the Hamiltonian number of graphs or of the classes of graphs which have been obtained in the past 30 years. We have already mentioned in Section 1.1 that the concept of the Hamiltonian number was introduced by Goodman and Hedetniemi [12] in 1973. Hamiltonian walks were studied further by Asano, Nishizeki, and Watanabe [2, 3], Bermond [4], and Vacek [21], Chartrand, Thomas, Zhang and Saenpholphat [7, 8, 20].

2.1 Basic Results on $h(G)$

Let us begin with the results of Goodman and Hedetniemi in [12] concerning the Hamiltonian number of graphs related to other graph parameters. Although the results presented in this section are simple and basic but they are useful and will be applied throughout our work.

The first result dealt with a relationship between the Hamiltonian number of graph and the Hamiltonian number of its certain subgraphs as stated in the following theorem.

Theorem 2.1.1 *Let G be a connected graph having blocks $B_1, B_2, \ldots, B_k$. Then*

$$h(G) = \sum_{i=1}^{k} h(B_i).$$

□

Theorem 2.1.1 has several implications, namely:

1. Every bridge of a graph G appears twice in every Hamiltonian walk of G.

2. If G is a tree of order n, then $h(G) = 2(n-1)$.

3. Let G be a connected graph having blocks $B_1, B_2, \ldots, B_k$. If $a_i \leq h(B_i) \leq b_i$ for all $i = 1, 2, \ldots, k$, then $a_1 + a_2 + \ldots + a_k \leq h(G) \leq b_1 + b_2 + \ldots + b_k$.

The next result gives a complete description for the Hamiltonian number of a complete n-partite graph as stated in the following theorem.

Theorem 2.1.2 *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph on $n_1 + n_2 + \ldots + n_k = n$ vertices, where $n_1 \leq n_2 \leq \ldots \leq n_k$. Then*

1. *G is Hamiltonian if and only if $n_1 + n_2 + \ldots + n_{k-1} \geq n_k$.*
2. *If $n_1 + n_2 + \ldots + n_{k-1} < n_k$, then $h(G) = 2n_k$.* □

It is usually difficult to obtain a good upper bound of $h(G)$. The following result provides an upper bound of $h(G)$ in terms of other easily computable parameters of a graph.

Theorem 2.1.3 *If G is a k-connected graph of order n having diameter d, then*

$$h(G) \leq 2n - \left\lfloor \frac{k}{2} \right\rfloor (2d - 2) - 2.$$

□

Determining a good lower bound for $h(G)$ tends to be more difficult than upper bounds. We first state a definition of *unicliqual* as a vertex of a graph that lies in only one clique.

Theorem 2.1.4 *Let U be the set of unicliqual vertices in G. Then*

$$h(G - U) + |U| \leq h(G).$$

□

In 1976, Bermond [4] obtained the following result.

Theorem 2.1.5 *Let G be a connected graph of order $n \geq 3$ and let k be an integer with $0 \leq k \leq n-2$. If $\deg(u)+\deg(v) \geq n-k$ every pair u, v of nonadjacent vertices of G, then $h(G) \leq n + k$.* □

2.2 A New Look at Hamiltonian Walks

Chartrand et. al [8] provided an alternative way to define $h(G)$ of a Hamiltonian walk in G. A Hamiltonian graph G contains a spanning cycle $C : v_1, v_2, \cdots, v_n$, $v_{n+1} = v_1$, where then $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n$. Thus Hamiltonian graphs of order $n \geq 3$ are those graphs for which there is a cyclic ordering $v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$ such that $\sum_{i=1}^{n} d(v_i, v_{i+1}) = n$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} for $1 \leq i \leq n$. For a connected graph G of order $n \geq 3$ and a cyclic ordering $s : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$, the number $d(s)$ is defined as

$$d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1}).$$

Therefore, $d(s) \geq n$ for each cyclic ordering s of $V(G)$. With this observation, it was shown in [8] that the Hamiltonian number $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of $V(G)$.

By the result of Theorem 2.1.1, if T is a tree of order n, then $h(T) = 2n - 2$. The following theorem shows that the converse of this statement is also true.

Theorem 2.2.1 *Let G be a connected graph of order n. Then*

$$h(G) = 2n - 2 \text{ if and only if } G \text{ is tree.}$$

□

The following result shows that every pair n, k of integers with $3 \leq n \leq k \leq 2n - 2$ is realizable as the Hamiltonian number as well as the order of some connected graph.

Proposition 2.2.2 *For every pair n, k of integers with $3 \leq n \leq k \leq 2n-2$, there exists a connected graph G of order n having $h(G) = k$.* □

In [8] the authors exploited the concept of a cyclic ordering of $V(G)$ of a connected graph G to define the *upper Hamiltonian number*, denoted by $h^+(G)$, as

$$h^+(G) = \max\{d(s)\},$$

where the maximum is taken over all cyclic orderings s of $V(G)$.

The exact values of $h^+(G)$ were obtained in many classes of graphs as stated in the following theorems.

Theorem 2.2.3 *Let Q_n be the hypercube of dimension $n \geq 2$. Then*

$$h^+(Q_n) = 2^{n-1}(2n-1).$$

□

Note that $h(G) \leq h^+(G)$ for every connected graph G. It was also shown in [8] that there are exactly two classes of connected graphs of order n for which their Hamiltonian number and upper Hamiltonian number are the same.

Theorem 2.2.4 *Let G be a connected graph of order n. Then*

$$h(G) = h^+(G) \text{ if and only if } G = K_n \text{ or } G = K_{1,n-1}.$$

□

Bounds for the upper Hamiltonian number in the class of trees and in the class of cycles were also discussed.

The Hamiltonian number and the upper Hamiltonian number of connected graphs were further discussed also in [7].

2.3 Graphs of Order n with Hamiltonian Number $2n-3$ or $2n-4$

It is well known that for a graph G of order n, $h(G) = 2n-2$ if and only if G is a tree. Saenpholphat and Zhang in [20] took a step further by characterizing all connected graph of order n with Hamiltonian number $2n-3$ or $2n-4$.

A connected graph with exactly one cycle is called a *unicyclic graph.* Let $\mathcal{U}_\mathcal{T}$ be the set of all unicyclic graphs containing exactly one triangle.

Theorem 2.3.1 *Let G be a connected graph of order $n \geq 3$. Then*

$$h(G) = 2n-3 \text{ if and only if } G \in \mathcal{U}_\mathcal{T}.$$

□

The following theorems shows a characterization of 2-connected graphs of order $n \geq 4$ with Hamiltonian number $2n-4$.

Theorem 2.3.2 *Let G be a connected graph of order $n \geq 4$. Then*

$$h(G) = 2n-4 \text{ if and only if } G \in \left\{K_4,\ K_{2,n-2},\ K_2 + \overline{K}_{n-2}\right\}.$$

□

Let $\mathcal{G}_1$ be the set of connected graphs G of order $n \geq 5$ with cut-vertices such that G contains exactly two blocks that are K_3 and each of the remaining blocks of G is K_2. Let $\mathcal{G}_2$ be the set of connected graphs G of order $n \geq 5$ with cut-vertices such that G contains exactly one block that is one of graphs in the set

$$S = \{K_4\} \cup \left\{K_{2,r-2}, K_2 + \overline{K}_{r-2} : 4 \leq r \leq n-1\right\}$$

and each of the remaining blocks of G is K_2.

Theorem 2.3.3 *Let G be a connected graph of order $n \geq 5$ with cut-vertices. Then*

$$h(G) = 2n-4 \text{ if and only if } G \in \mathcal{G}_1 \cup \mathcal{G}_2.$$

□

It has shown that the two upper bounds for the Hamiltonian number of a connected graph can be in terms of (1) its order and clique number and (2) its order and connectivity.

The *clique number* $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. The following result shows an upper bound for $h(G)$ in terms of the order and clique number of a connected graph G.

Proposition 2.3.4 *If G is a nontrivial connected graph of order n having clique number ω, then*

$$h(G) \leq 2n - \omega.$$

Furthermore, for each integer ω with $2 \leq \omega \leq n$, there exists a connected graph F of order n having clique number ω such that $h(F) = 2n - \omega$. □

Let $\kappa(G)$ be the connectivity of a graph G. It is known that $\kappa(G) \le \delta(G)$ for every graph G. A graph G is *k-connected* if $\kappa(G) \ge k$.

Proposition 2.3.4 *Let G be a nontrivial k-connected graph of order n and diameter $d \ge 3$. If G is not Hamiltonian, then $h(G) \le 2(n-k)$.* □

Observe that if G is a non-Hamiltonian k-connected graph of order n and diameter 2, then $h(G) \le 2n - k - 1$. The authors posted the following conjecture.

Conjecture 2.3.6 *Let G be a k-connected graph of order n and diameter 2. If G is not Hamiltonian, then $h(G) \le 2n - 2k$.* □

CHAPTER 3

INTERPOLATION AND EXTREMAL THEOREMS: THE HAMILTONIAN NUMBER OF CUBIC GRAPHS

This chapter is to present our comprehensive work concerning interpolation and extremal results for the graph parameter h. It consists of 4 sections. Section 3.1 deals with our first results concerning the Hamiltonian number of the generalized Petersen graphs and some other related classes of cubic graphs. We introduce the concept of an i-Hamiltonian graph as a connected graph G of order n having $h(G) = n + i$. In section 3.1, we characterize all connected graphs G of order n where $h(G) = n+1$. We continue our investigation for the Hamiltonian number in the class of connected cubic graphs of order n in Section 3.2. We have successfully determined the range of Hamiltonian numbers in the class of connected cubic graphs, $\mathcal{CR}(3^n)$. It should be noted that for even integers $n \neq 14$, we have that the range of Hamiltonian numbers in the class of connected cubic graphs $h(3^n)$ completely covers all integers from n to $\max(h, 3^n) = \max\{h(G) : G \in \mathcal{CR}(3^n)\}$. We found that cubic graphs G of order n with $h(G) = \max(h, 3^n)$ are those graphs which contain as many cut edges as possible. With these observation we consider the problem of determining the range of Hamiltonian numbers in the class of 2-connected cubic graphs of order n. This problem was also motivated by a result of Robinson and Wormald [16]. They proved by using a probabilistic method that if H is the number of Hamiltonian cycles in a cubic graph chosen uniformly at random from all labelled cubic graphs on $2n$ vertices, then

$$\lim_{n\to\infty} Pr(H > 0) = 1.$$

We produce some classes of cubic graphs of order $2n$ which are far from being Hamiltonian. Details can be found in Section 3.3. It is not difficult to believe that a connected graph G with small circumference may have a large Hamiltonian number. On the other hand, a connected graph having a higher connectivity may have a small Hamiltonian number. It was proved by Chavátal and Erdős [9] that a graph G with at least three vertices and $\kappa(G) \geq \beta(G)$ is Hamiltonian. This result suggests us to consider the problem of determining the range of Hamiltonian numbers in the class of graphs G of order n and $\kappa(G) = k$. Both interpolation and extremal results are obtained in all situations. Furthermore, we are able to use our result to prove a conjecture posted by Saenpholphat and Zhang [20]. Details can be found in Section 3.4.

3.1 Almost Hamiltonian Cubic Graphs

A connected graph G of order n is called an *i-Hamiltonian graph* if $h(G) = n+i$. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an *almost Hamiltonian graph.*

Let $P(k,m)$ be a generalized Petersen graph such that $V(P(k,m)) = \{u_i, v_i : i = 0, 1, \cdots, k-1\}$ and $E(P(k,m)) = \{u_iu_{i+1}, v_iv_{i+m}, u_iv_i : i = 0, 1, \cdots, k-1\}$ where addition is taken modulo k and $m \leq \frac{k}{2}$. The graph $P(5,2)$ is the Petersen graph. In [1] Alspach completely determined all integers k and m in which $P(k,m)$ is Hamiltonian as we will state in the following theorem.

Theorem 3.1.1 *The generalized Petersen graph $P(k,m)$ is non-Hamiltonian if and only if $m = 2$ and $k \equiv 5 (\text{mod } 6)$.* □

Since $P(k,m)$ is a graph of order $2k$, it follows that $h(P(6t-1,2)) \geq 2(6t-1)+1 = 12t-1$ for all integers $t \geq 1$. Next theorem we show that $P(k,m)$ is an almost Hamiltonian graph if and only if $m = 2$ and $k \equiv 5(\text{mod } 6)$.

Theorem 3.1.2 *Let $P(k,m)$ be a generalized Petersen graph. Then*

$$h(P(k,m)) = \begin{cases} 2k+1 & \textit{if } m = 2 \textit{ and } k \equiv 5(\text{mod } 6), \\ 2k & \textit{otherwise.} \end{cases}$$

Proof. By Theorem 3.1.1, it is enough to produce a closed spanning walk of $P(k,m)$ of length $2k+1$ for $m = 2$ and $k \equiv 5(\text{mod } 6)$. Define W be defined as follows

$$W : v_0, v_2, \cdots, v_{k-1}, v_1, v_3, \cdots, v_{k-2}, u_{k-2}, u_{k-3}, u_{k-4}, \cdots, u_1, u_0, u_{k-1}, u_0, v_0.$$

It is clear that W has length $2k+1$. Thus $h((P(k,m)) = 2k+1$. □

It was shown in [19] that all connected cubic graphs of order n, where $4 \leq n \leq 8$, are Hamiltonian. It was also shown in [19] that the Petersen graph $P(5,2)$ and the Tietze graph (denoted by T_{12}) are the only 2-connected cubic graph of order 10 and 12, respectively, that are not Hamiltonian. They are, in fact, almost Hamiltonian cubic graphs of respective order. Note that T_{12} is obtained from $P(5,2)$ by replacing one vertex of $P(5,2)$ with a triangle and joining the vertices of the triangle to its former neighbors of the replaced vertex (see Figure 4).

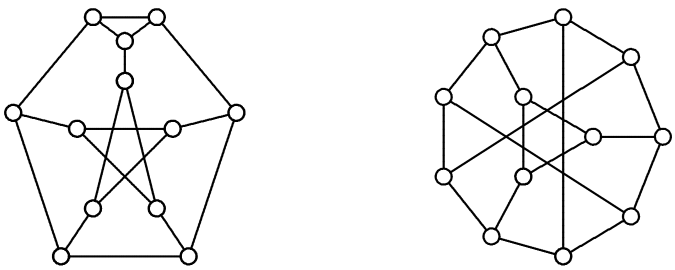


Figure 4 The Tietze Graph T_{12}.

Let G be a cubic graph and $v \in V(G)$. We denote $G * v$ to be the graph obtained from G by replacing v by a triangle and joining the vertices of the triangle to the former neighbors of v as shown in Figure 5. Thus $G * v$ is also a cubic graph containing a triangle.

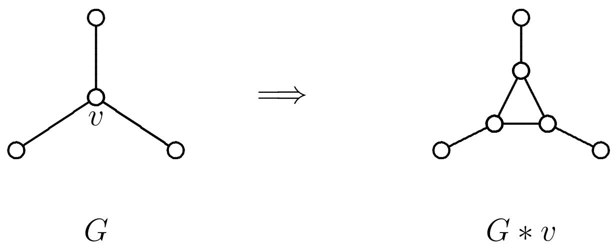


Figure 5 Part of graph G and $G * v$.

Theorem 3.1.3 *Let G be a cubic graph of order $n \geq 4$ and $v \in V(G)$. Then G is Hamiltonian if and only if $G * v$ is Hamiltonian.*

Proof. Let G be a cubic graph and $V(G) = \{v_1, v_2, \cdots, v_n\}$. Put $v = v_1$. Thus $G * v$ is the graph with $V(G * v) = (V(G) - v) \cup \{x_1, y_1, z_1\}$, $\{x_1, y_1, z_1\}$ induced a triangle in $G * v$ and $y_1v_2, v_nz_1 \in E(G * v)$.

Suppose that G is Hamiltonian. Without loss of generality we may assume that

$$C : v_1, v_2, \cdots, v_n, v_1$$

is a Hamiltonian cycle of G. Thus

$$C_v : z_1, x_1, y_1, v_2, v_3, \cdots, v_n, z_1$$

is a Hamiltonian cycle of $G * v$.

Conversely, suppose that $G * v$ is Hamiltonian and let

$$C_v : u_1, u_2, \cdots, u_{n+2}, u_1$$

be a Hamiltonian cycle of $G * v$. If x_1 is not a neighbor of y_1 and z_1 in C_v, then $d_{G*v}(x_1) \geq 4$. Thus x_1 is a neighbor of y_1 or z_1 in C_v. It is also true for y_1 and z_1. Thus x_1, y_1, z_1 must appear as consecutive vertices in C_v. Deleting the three vertices and replacing them by v_1, we obtain a Hamiltonian cycle of G. □

Let G be a cubic graph of order n with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Put $G^1 = G * v_1$ and put $G^{i+1} = G^i * v_{i+1}$ for an integer i, $1 \leq i \leq n-1$. Thus from Theorem 3.1.3 we have the following corollary.

Corollary 3.1.4 *Let G be a cubic graph of order n. Then G is Hamiltonian if and only if G^i is Hamiltonian for all $1 \leq i \leq n$.* □

By Theorem 3.1.3, we also have that if G is not Hamiltonian cubic graph, then $G * v$ is not Hamiltonian.

The *subdivision graph* of a graph G is a graph obtained from G by deleting an edge uv of G and replacing it by a vertex w of degree 2 that is joined to u and v. Let K_4^+ be a subdivision of K_4 obtained by inserting a vertex of degree 2 into an edge of K_4 (see Figure 6).

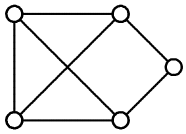


Figure 6 Graph K_4^+.

Let G be a graph obtained from three copies of K_4^+ and a new vertex v by connecting three vertices of degree two of K_4^+ to v (see Figure 7). Thus G is a cubic of order 16 with $h(G) = 21$ but $h(G * v) = 24$. It is clear that G is a 5-Hamiltonian and $G * v$ is 6-Hamiltonian. Note that $G * w$ is 5-Hamiltonian, for each vertex w of G which is not v.

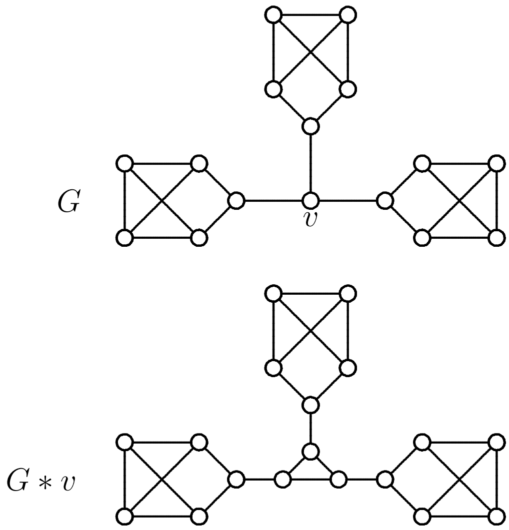


Figure 7 Graphs G and $G * v$.

Theorem 3.1.5 *For an even integer $n \geq 10$, there exists an almost Hamiltonian cubic graph of order n.*

Proof. The Petersen graph $P(5, 2)$ is the unique almost Hamiltonian cubic graph of order 10 and the Tietze graph T_{12} is also the unique almost Hamiltonian cubic graph of order 12 and $T_{12} = G * v$, where $G = P(5, 2)$ and $v \in V(P(5, 2))$. Let u_1, v_1, w_1 be the induced triangle of T_{12} and $T_{14} = T_{12} * v_1$ (see Figure 8). Thus for an integer $i \geq 1$, let u_i, v_i, w_i be the induced triangle of $T_{12+2(i-1)}$ and $T_{12+2i} = T_{12+2(i-1)} * v_i$. By assuming that the graph $T_{12+2(i-1)}$ is almost Hamiltonian, we have that $h(T_{12+2i}) \leq 12 + 2i + 1$. By Theorem 3.1.3 we have that T_{12+2i} is not Hamiltonian. Therefore $h(T_{12+2i}) = 12 + 2i + 1$ and T_{12+2i} is almost Hamiltonian.

□

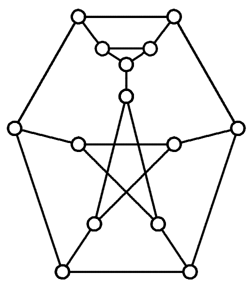


Figure 8 Graph T_{14}.

A Hamiltonian graph is necessarily 2-connected. The same result also holds in the class of almost Hamiltonian cubic graphs. The following Theorem 3.1.7 can be considered as a characterization of cubic graphs for being almost Hamiltonian graph, but first we show that an almost Hamiltonian cubic graph is also 2-connected.

Theorem 3.1.6 *Let G be a connected cubic graph of order $n \geq 10$. If G is almost Hamiltonian, then G is 2-connected.*

Proof. Suppose that G is not 2-connected and v is a cut vertex of G. Since G is cubic, there exists a vertex u such that u is also a cut vertex of G and u is adjacent to v. Furthermore, uv is a cut edge of G. Let $G - e = G_1 \cup G_2$. It follows that $h(G) \geq h(G_1) + h(G_2) + 2 \geq n + 2$. The proof is complete. □

Theorem 3.1.7 *Let G be a connected non-Hamiltonian cubic graph of order $n \geq 10$. Then G is an almost Hamiltonian graph if and only if for every Hamiltonian walk W of G, W contains a cycle of order $n - 1$.*

Proof. Suppose that $h(G) = n + 1$. Let $v_1, v_2, \cdots, v_{n+2} = v_1$ be a Hamiltonian walk of length $n+1$. Thus there exist v_i and v_j with $1 \leq i < j \leq n$ and $v_i = v_j$ and all other vertices are distinct. Without loss of generality we may assume that $i = 1$. If $j \geq 4$, then $d(v_1) \geq 4$. Thus for $j = 3$, $v_3, v_4, \cdots, v_{n+2} = v_3$ is a cycle in G of length $n - 1$. Conversely, suppose G contains a cycle $v_1, v_2, \cdots, v_{n-1} = v_1$ of length $n - 1$. Let $v \in V(G) - \{v_1 = v_{n-1}, v_2, v_3, \cdots, v_{n-1}\}$. Thus there exists an integer k with $1 \leq k \leq n - 1$ such that v_k is adjacent to v_n. We now form a Hamiltonian walk $v_1, v_2, \cdots, v_k, v, v_k, \cdots, v_{n-1} = v_1$ and this walk has length $n + 1$. Therefore $h(G) = n + 1$. □

Let G be a Hamiltonian cubic graph. We have shown in Theorem 3.1.3 that for every $v \in V(G)$, $G{*}v$ is Hamiltonian and vise versa. We have also mentioned that there is a 5-Hamiltonian graph G and $v \in V(G)$ such that $G * v$ is 6-Hamiltonian.

Let G be a connected cubic graph of order n with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Denote that $G^* = G^n$. Figure 9 shows the graph $P^*(5, 2)$.

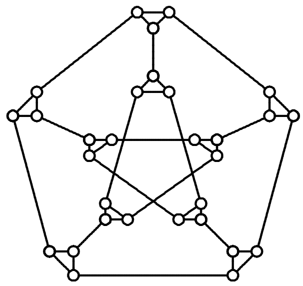


Figure 9 Graph $P^*(5, 2)$.

Theorem 3.1.8 *If G is an almost Hamiltonian cubic graph of order n, then $h(G^*) = 3n + 2$.*

Proof. By Theorem 3.1.3, it follows that $h(G^*) \geq 3n + 1$. Assume, to the contrary, that $h(G^*) = 3n + 1$. By Theorem 3.1.7, let $C : x_1, x_2, \cdots, x_{3n-1}, x_1$ be a cycle of length $3n - 1$ of G^*, where $V(G^*) = \{x_1, x_2, \cdots, x_{3n}\}$. Without loss of generality we may assume that x_{3n} is adjacent to x_1. Since G^* is non-Hamiltonian, $x_{3n}x_2, x_{3n}x_{3n-1} \notin E(G^*)$. Since G^* is cubic, there exist i, j with $1 < i < j < 3n - 1$ such $\{x_{3n}, x_i, x_j\}$ induced a triangle in G^* and also $j = i + 1$. Therefore G^* is Hamiltonian. This is a contradiction.

In order to show that $h(G^*) = 3n + 2$, we will construct a Hamiltonian walk of G^* of length $3n + 2$. As in the proof of Theorem 3.1.7, let

$$W : v_1, v_2, \cdots, v_k, v, v_k, \cdots, v_n = v_1$$

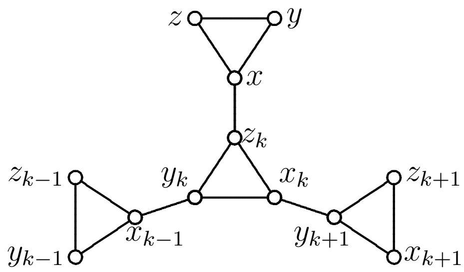


Figure 10 A Part of Graph G^*.

be a Hamiltonian walk of G. For each i, $1 \leq i \leq n$, we replace vertices v_i and v in W by triangles x_i, y_i, z_i and x, y, z respectively, and then arrange them in such a way that x_i is adjacent to y_{i+1}, for all $i = 1, 2, \cdots, n-1$. Without loss of generality we may assume that z_k is adjacent to x and vertices x_k, y_k, z_k are arranged as shown in Figure 10. Thus the Hamiltonian walk
$W : z_1, x_1, y_2, z_2, x_2, \cdots, y_{k-1}, z_{k-1}, x_{k-1}, y_k, z_k, x, y, z, x, z_k, x_k, y_{k+1}, z_{k+1}, x_{k+1}, \cdots,$
$y_{n-1}, z_{n-1}, x_{n-1}, y_n, z_n = z_1$ has length $3n + 2$. □

The following result can be obtained as a direct consequence of Theorem 3.1.8.

Corollary 3.1.9 *$h(P^*(k, 2)) = 6k + 2$, for every positive integer k with $k \equiv 5 \pmod 6$.* □

3.2 The Hamiltonian Number of Cubic Graphs

We have already seen several results concerning Hamiltonian numbers for some specific classes of cubic graphs obtained in Section 3.1. Now we solve the following as explained in the beginning of this chapter problem. We will see that several results concerning the structure of cycles in 2-connected cubic graphs will be used throughout the proof.

Problem Let $\mathcal{CR}(3^n)$ be the class of connected cubic graphs of order n and

$$h(3^n) = \{h(G) : G \in \mathcal{CR}(3^n)\}.$$

Find $h(3^n)$.

The following graph constructions and notation will be used in this section from now on.

1. Let $\mathcal{CR}(3^n)$ be the set of all connected cubic graphs of order n. For $n \geq 10$, let $\mathcal{CR}_1(3^n)$ be the class of connected cubic graphs of order n containing a cut edge and $\mathcal{CR}_2(3^n)$ the class of 2-connected cubic graphs.

2. Let G be a cubic graph and $v \in V(G)$. The cubic graph $G * v$ has already been defined earlier.

3. Let G be a cubic graph. We denote G^+ a subdivision of G obtained by inserting a vertex of degree 2 into an edge of G. Thus K_4^+ is unique (see Figure 6). When the inserted vertex in a subdivision of G is specified, say u, we denote $G(u)$ a graph with $V(G(u)) = V(G) \cup \{u\}$ and $E(G(u)) = (E(G) - xy) \cup \{xu, uy\}$, where $xy \in E(G)$. For the graph K_4 of Figure 11, the graph $K_4(u)$ is shown.

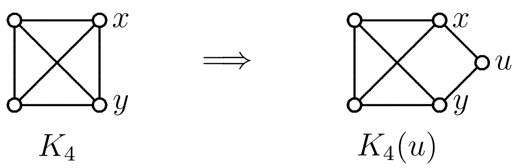


Figure 11 Graph $K_4(u)$.

4. Let G and H be vertex disjoint cubic graphs. Let u, v be new vertices. We denote $\langle G(u), H(v)\rangle$ a connected cubic graph of minimum order containing $G(u) \cup H(v)$ as its induced subgraph. Note that the graphs $G(u)$ and $H(v)$

are not unique but the graph $\langle G(u), H(v)\rangle$ is uniquely determined by $G(u)$ and $H(v)$ and $|\langle G(u), H(v)\rangle| = |G(u)| + |H(v)|$.

5. Let G, H and K be pairwise vertex disjoint cubic graphs. Let x, y, z be new vertices. A connected cubic graph of minimum order containing $G(x) \cup H(y) \cup K(z)$ as its induced subgraph is denoted by $\langle G(x), H(y), K(z)\rangle$. Note that the graph $\langle G(x), H(y), K(z)\rangle$ is uniquely determined by $G(x), H(y)$ and $K(z)$. Then $|\langle G(x), H(y), K(z)\rangle| = |G(x)| + |H(y)| + |K(z)| + 1$. More generally, if $G_1, G_2, \ldots, G_k$ are pairwise disjoint graphs and for all $i = 1, 2.\ldots, k$, vertices of G_i are of degree 2 or 3 with at least one vertex of degree 2, then $\langle G_1, G_2, \ldots, G_k\rangle$ denotes a connected cubic graph of minimum order containing $G_1 \cup G_2 \cup \ldots \cup G_k$ as its subgraph.

6. We denote K_4^- the graph obtained from K_4 by removing an edge.

3.2.1 Cycles in 2-Connected Cubic Graphs

A *factor* of a graph G is a spanning subgraph of G. A *k-factor* of a graph G is a k-regular spanning subgraph of G. In particular, a 1-factor of a graph G is a 1-regular spanning subgraph of G and a 2-factor is a 2-regular spanning subgraph of G. Let G be a connected cubic graph containing a 1-factor F_1 and let F be a graph with $V(F) = V(G)$ and $E(F) = E(G) - E(F_1)$. Then F is a 2-factor of G.

By a well-known theorem of Petersen [15], every 2-connected cubic graph G has a 2-factor. Thus if G is a 2-connected cubic graph, then the edge set of G can be partitioned into a 1-factor and a 2-factor. The following theorem due to Schönberger [18] and it is considered as an extension of the Petersen theorem.

Theorem 3.2.1 *Let G be a 2-connected cubic graph and $e \in E(G)$. Then G has a 1-factor containing e.* □

As a consequence of Theorem 3.2.1, if G is a 2-connected cubic graph and e, f are two incident edges of G, then G has a 2-factor containing both e and f.

In [19], it was shown that there are two 2-connected cubic graphs of order

6 and five 2-connected cubic graphs of order 8 as shown in Figure 12 and Figure 13. All of those graphs are Hamiltonian.

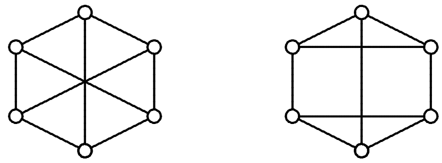


Figure 12 Two 2-connected Cubic Graphs of Order 6.

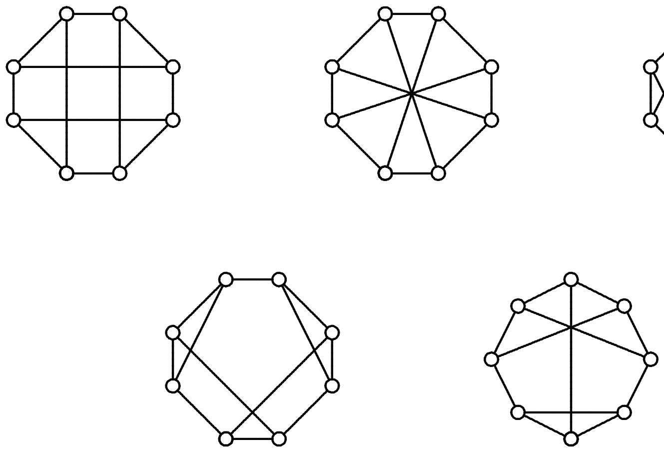


Figure 13 Five 2-connected Cubic Graphs of Order 8.

For an integer $m \geq 3$, let C_m denote an m-cycle, a cycle of order m. The length of a smallest cycle in a graph is referred to as its *girth*.

Let G be a 2-connected cubic graph of order $n \geq 10$ and $F = \bigcup_{i=1}^{k} C_{p_i}$ be a 2-factor of G. If the girth of G is at least 5, then for each $i = 1, 2, \ldots, k$, $p_i \geq 5$.

Let $T = \{x, y, z\}$ be a triangle of a cubic graph G. Then T is called a *pure triangle* if x, y, z have no other neighbor in common. If G contains a pure triangle $T = \{x, y, z\}$, then we define G' to be the graph obtained from G by replacing the vertices x, y, z by a new vertex v, and joining v to the third neighbors of x, y and z as shown in Figure 14. Thus G' is a 2-connected simple cubic graph of order $n - 2$.

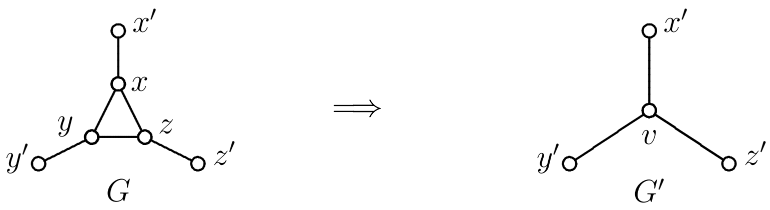


Figure 14 The Graph G'.

Theorem 3.2.2 *Let G be a 2-connected cubic graph of order $n \geq 6$. Then there exists a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ of G such that for all $i = 1, 2, \ldots, k$, $p_i \geq 4$. Moreover, if $n = 4q$, for some integer q, then $k < \frac{n}{4}$.*

Proof. Let G be a 2-connected graph of order $n \geq 6$. The result trivially holds if the girth of G is at least 5. Therefore we assume that the girth of G is at most 4.

If $n = 6$, then G is Hamiltonian. Suppose that G contains a pure triangle $T = \{x, y, z\}$. Let x', y', z' be the third neighbors of x, y, z, respectively (see Figure 14). Then we define G' to be the graph obtained from G by replacing the vertices x, y, z by a new vertex v, joining v to the neighbors of x, y, z not in $V(T)$. Thus G' is 2-connected simple cubic graph of order $n - 2$. By induction, there is a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ of G' such that for all $i = 1, 2, \ldots, m$, $q_i \geq 4$. Let C' be the cycle in F' containing v and suppose without loss of generality that C' contains x', v, z' as its consecutive vertices. By replacing v by x, y, z yields a cycle C of G. Thus we obtain a 2-factor $F = (F' - \{C'\}) \cup \{C\}$ of G satisfying the desired property.

Since K_4^- consists of two triangles with a common edge, any triangle of K_4^- can not belong to any 2-factor of G. Therefore, if G does not contain a pure triangle, then any 2-factor of G is a union of cycles of length at least 4.

Suppose that $n = 4q$, for some integer q. Suppose further that G does not contain a pure triangle and all 2-factors of G is a union of 4-cycles. Let $F = \bigcup_{i=1}^{k} C_{q_i}$ be a 2-factor of G such that $q_1 = q_2 = \ldots = q_k = 4$. Let C be a 4-cycle in F, $V(C) = \{x, y, z, w\}$, $E(C) = \{xy, yz, zw, wx\}$ and x', y', z', w' are the third neighbors of x, y, z, w, respectively as shown in Figure 15. If $xz \notin E(G)$ and $yw \notin E(G)$, then xx', yy', zz', ww' are independent and hence x', y', z', w' are pairwise distinct. By

Theorem 3.2.1, let F' be a 2-factor of G containing $x'x, xy$. Then, by assumption that all cycles in F' are 4-cycles, either $\{x', x, y, y'\}$ or $\{x', x, y, z\}$ induces a 4-cycle in G. Since $x' \neq z'$, it follows that $\{x', x, y, y'\}$ induces a 4-cycle in G. Similarly, each of $\{z', z, w, w'\}, \{x', x, w, w'\}, \{y', y, z, z'\}$ also induces a 4-cycle in G. Thus G is a Hamiltonian graph of order 8. Suppose that all 4-cycles in F induce K_4^- in G. Then G is Hamiltonian. Therefore, G has a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ of G with $k < \frac{n}{4}$, as required. □

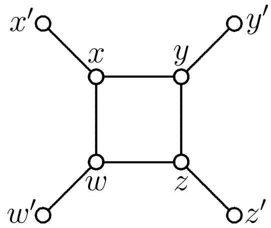


Figure 15 A Part of Cycle C in 2-factor F of G'.

By using the result in Theorem 3.2.1, we obtain the following stronger result:

Theorem 3.2.3 *Let G be a 2-connected cubic graph of order $n \geq 6$ and $e \in E(G)$. Then G has a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ such that for all $i = 1, 2, \ldots, k$, $p_i \geq 4$ and $e \notin E(F)$. Moreover, if $n = 4q$, for some integer q, then $k < \frac{n}{4}$.*

Proof. By Theorems 3.2.1 and 3.2.2, the result of this theorem holds where G does not contain a pure triangle as its subgraph. The result also holds if $n = 6, 8$. Suppose that G contains a pure triangle $T = \{x, y, z\}$, $n \geq 10$, and let G' be the graph as described earlier in Theorem 3.2.2. Thus G' is of order $n - 2$. By induction, for every edge $e \in G'$, G' has a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ such that for all $i = 1, 2, \ldots, m$, $q_i \geq 4$ and $e \notin E(F')$. Let C' be the cycle in F' containing v. If $e \notin \{vx', vy', vz'\}$, then, without loss of generality, we may assume that y', v, z' are consecutive vertices in C'. Let C be a cycle obtained from C' by replacing v by y, x, z. Thus $F = (F' - \{C'\}) \cup \{C\}$ is a 2-factor of G with the desired property. If $e \in \{vx', vy', vz'\}$, say $e = vx'$, then y', v, z' are consecutive vertices on C'. Let C be a cycle obtained from C' by replacing v by y, x, z. Thus $F = (F' - \{C'\}) \cup \{C\}$ is a 2-factor of G with the desired property. If $e \in \{xy, yz, xz\}$, say $e = yz$, then there exists a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ of G' for all $i = 1, 2, \ldots, m$, $q_i \geq 4$, $vx' \notin E(F')$. Thus there exists a cycle C' in F' such that C' contains y', v, z' as its consecutive vertices. Thus we can extend F' to a 2-factor F of G as described above. Thus the proof is complete. $\square$

3.2.2 The Range of the Hamiltonian Numbers

For an even integer $n \geq 4$, we have already denoted that $h(3^n) = \{h(G) : G \in \mathcal{CR}(3^n)\}$. We put $\min(h, 3^n) = \min\{h(G) : G \in \mathcal{CR}(3^n)\}$ and $\max(h, 3^n) = \max\{h(G) : G \in \mathcal{CR}(3^n)\}$. It is well-known that for any even integer $n \geq 4$, there exists a Hamiltonian cubic graph of order n. Thus $\min(h, 3^n) = n$. It is also well-known that $\max(h, 3^n) = \min(h, 3^n) = n$ if and only if $n = 4, 6, 8$. For $n = 10, 12$, we have $\max(h, 3^n) = n+2$. Observe that for $n = 10, 12$, a connected cubic graph G having $h(G) = \max(h, 3^n)$ is a graph with a cut edge. The problem of determining $\max(h, 3^n)$ is more challenging. The following facts are useful.

1. If a connected graph G contains an edge e such that $G - e$ is connected, then $h(G) \leq h(G - e)$.

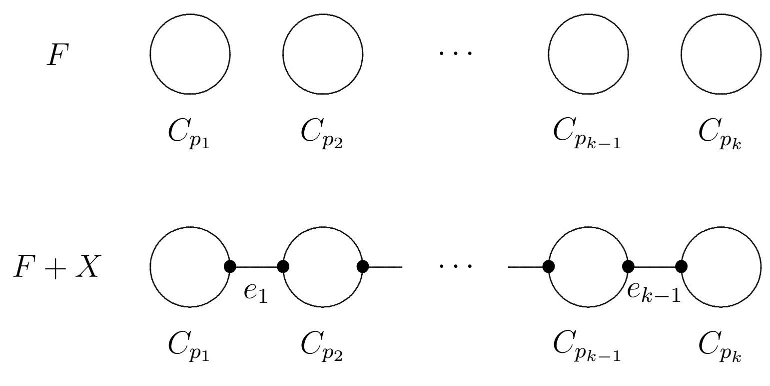


Figure 16 The Construction of the Graph $F + X$.

2. If G is a 2-connected cubic graph of order n and $F = \bigcup_{i=1}^{k} C_{p_i}$ is a 2-factor of G, then there exists a set $X = \{e_1, e_2, \ldots, e_{k-1}\} \subseteq E(G) - E(F)$ such that $F + X$ is connected. Thus $h(G) \leq h(F+X) = n+2(k-1)$. In particular, if G is a 2-connected cubic graph of order n, then, by Theorem 3.2.2, $h(G) \leq n+2(k-1)$, where $k \leq \lfloor \frac{n-2}{4} \rfloor$.

3. For an integers q and n, $n \geq 10$, let $H = \langle (q-2)K_4^-, 2K_4^+ \rangle$ for $n = 4q+2$ (see Figure 17) and $H = \langle (q-2)K_4^-, K_4^+, K \rangle$ for $n = 4(q+1)$ (see Figure 18), where K is a graph obtained from a subdivision of cubic graph of order 6 and a subdivision of an edge. Then, by Theorem 2.1.1, $h(H) = n + 2(q-1)$.

Figure 17 The Graph H of order $4q + 2$.

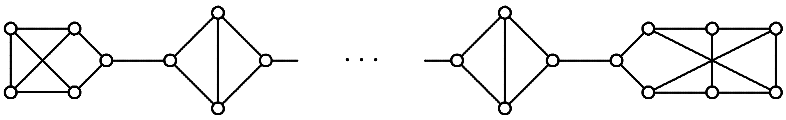


Figure 18 The Graph H of order $4(q + 2)$.

As a consequence of above observations, we obtain the following lemma.

Lemma 3.2.4 *Let G be a 2-connected cubic graph of order $n \geq 10$. Then there exists a graph $H \in \mathcal{CR}_1(3^n)$ such that $h(G) \leq h(H)$.*

Proof. By using above observation and Theorem 3.2.2, we have $k \leq q$. Thus $h(G) \leq n + 2(k-1) \leq n + 2(q-1) = h(H)$. Note that $H \in \mathcal{CR}_1(3^n)$. □

By Theorem 3.2.4 and Lemma 3.2.3, the following corollary can be easily obtained.

Corollary 3.2.5 *Let G be a 2-connected cubic graph of order $n \geq 10$ and $e = xy \in E(G)$. Then there exist a closed spanning walk W of G such that x, y appear as consecutive vertices on W exactly once and a graph $H \in \mathcal{CR}_1(3^n)$ such that $|W| \leq h(H)$.* □

The result of Lemma 3.2.4 suggests that the value of $\max(h, 3^n)$ can be obtained from a cubic graph of order n having as many cut edges as possible. Let $G \in \mathcal{CR}_1(3^n)$ with $h(G) = \max(h, 3^n)$. Let $B_1, B_2, \ldots, B_k$ be the blocks of G with $|B_i| \geq 3$, for all $i = 1, 2, \ldots, k$. Thus there are $k-1$ blocks of G of order 2. A block of G of order 2 is called *trivial*, otherwise, it is called *nontrivial*. A nontrivial block B is called a *leaf block* if B has exactly one vertex of degree 2. It is clear that G has at least two leaf blocks. Thus for a nontrivial block B, we mean in this paper, a 2-connected graph of order at least 3, $2 \leq d(x) \leq 3$, for all $x \in V(B)$, and B contains at least one vertex of degree 2. Similarly, by a leaf block, we mean a nontrivial block with exactly one vertex of degree 2.

Lemma 3.2.6 *Let B be a leaf block of order b and $5 \leq b \leq 9$. Then B is Hamiltonian.*

Proof. Since B is a leaf block of order b, b is odd and $b \geq 5$. Let v be the vertex of degree 2 in $V(B)$. Then B can be shown as in Figure 19. Let $B_1 = B_1' + u_r v_r$. Then B_1 is a 2-connected cubic graph of order at most 8 and hence B_1 is Hamiltonian. As consequently Theorem 3.2.1, B_1 has a Hamiltonian cycle C' containing $u_r v_r$. The cycle C' can be extended to a Hamiltonian cycle C of B. Thus B is Hamiltonian.

□

Let B be a nontrivial block. If $W : x_1, x_2, \ldots, x_{t+1} = x_1$ is a closed spanning walk of B and $e = xy \in E(B)$, then we say that x, y appear as consecutive vertices in W if there exists i, $1 \leq i \leq t$, such that $\{x, y\} = \{x_i, x_{i+1}\}$. If x, y, x appear in W as consecutive vertices, then we say that x, y appear twice in W.

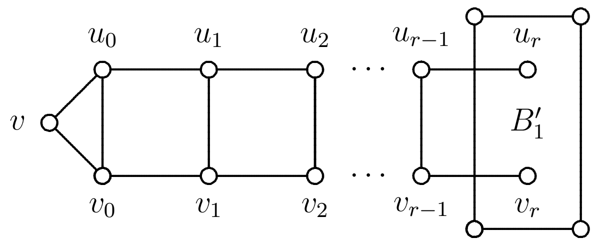


Figure 19 A Leaf Block B of Graph G.

Lemma 3.2.7 *Let B be a leaf block of order $b \geq 11$. Then there exist a closed spanning walk $W : x_1, x_2, \ldots, x_{t+1} = x_1$ of B and a connected graph H with a cut edge such that B and H are of the same degree sequence, and $|W| \leq h(H)$.*

Proof. Let B be a leaf block of G of order $b \geq 11$ as shown in Figure 19. If B is Hamiltonian of order $b \geq 11$, then we can choose a Hamiltonian walk W of B and a graph H obtained from a connected cubic graph of order $b-1$ with a subdivision the cut edge e. Thus $|W| \leq h(H)$ and H and B have the same degree sequence. Suppose that B is not Hamiltonian. Let $B_1 = B_1' + u_r v_r$. Thus B_1 is 2-connected cubic graph of order $b_1 \leq b - 1$. If $b_1 \leq 8$, then B is Hamiltonian. Suppose that $b_1 \geq 10$. By Corollary 3.2.5, there exists a closed spanning walk W_1 of B_1 such that u_r, v_r appear as consecutive vertices in W_1 by exactly once and a graph $H_1 \in \mathcal{CR}_1(3^n)$ such that $|W_1| \leq h(H_1)$, where B_1 and H_1 have the same degree sequence. Thus W_1 and H_1 can be easily extended to W and H with desired result. The proof is complete. □

For a 2-connected cubic graph G, we have constructed a closed spanning walk W of G traveling along the cycles in a given 2-factor of G and $k-1$ edges connecting between k cycles in the 2-factor. It turns out that for every edge on the k cycles appears exactly once in W while every edge that connects to the cycles appears exactly twice in W. Lemma 3.2.6 showed the similar result for a leaf block.

Let B be a block of order $b \geq 3$. If $b = 3, 4$, then B is Hamiltonian. If $b = 5$ and B is not Hamiltonian, then $B \cong K_{2,3}$ and $h(B) = 6$. Note that the graph $K_{2,3}$ has a property that for every edge $e = xy$, there is a Hamiltonian walk containing

x, y as its consecutive vertices exactly once.

If $b \in \{6, 8\}$, B contains exactly two vertices of degree 2, and the two vertices x, y are not adjacent, then the graph $B' = B + xy$ is a cubic graph of order $b \in \{6, 8\}$. Thus B' is Hamiltonian. By Theorem 3.2.3, B' has a Hamiltonian cycle not containing xy. Thus B is Hamiltonian. Suppose that the two vertices are adjacent. Let B' be the graph obtained from B by removing the two vertices of degree 2. Thus $B' \cong K_4^-$ if $b = 6$, and $B' \cong K$ if $b = 8$, where K is obtained from a cubic graph of order 6 by removing one edge. Thus B' is Hamiltonian and consequently B is Hamiltonian.

If $b \in \{7, 9\}$, B contains exactly three vertices of degree 2, then the three vertices are not pairwise adjacent. Thus there are two vertices x, y of degree 2 in B that are not adjacent and $B' = B + xy$ is a leaf block of order $b \in \{7, 9\}$. By Lemma 3.2.6, B' is Hamiltonian.

Theorem 3.2.8 *Let B be a nontrivial block of order $b \geq 6$ of a 2-connected cubic graph. Then B is Hamiltonian or there exist a closed spanning walk W of B and a connected graph H with a cut edge satisfying the following conditions.*

1. *For every edge xy of B, x, y appear as its consecutive vertices in W by at most twice,*

2. *The H and the block B have of the same degree sequence, and*

3. *$h(H) \geq |W|$.*

Proof. Let B be a nontrivial block of order $b \geq 6$. Suppose that B is a non-Hamiltonian graph. If $b = 6$, then B contains exactly four vertices of degree 2 and $h(B) = 7$. Let H be a graph of order 6 obtained from two disjoint triangles and one edge joining from one vertex of a triangle to one vertex of the other triangle. Thus H contains a cut edge and $h(H) = 8$. Thus the result follows if $b = 6$.

Suppose that $b \geq 7$. Let B' be the subgraph of B induced by $\{v, u_0, v_0, u_1, v_1, \ldots, u_{r-1}, v_{r-1}\}$ as shown in Figure 19. Thus $V(B) = V(B') \cup V(B'_1)$ and $E(B) = E(B') \cup E(B'_1) \cup \{u_{r-1}u_r, v_{r-1}v_r\}$. Let $B_1 = B'_1 + u_rv_r$. Then B_1 is a block of order at most $b-1$. If B_1 is of order 5, then either B_1 is Hamiltonian or $B_1 \cong K_{2,3}$. Since every block of order 5 has a property that every edge is contained in its closed spanning walk of length 6, it follows that $h(B) \leq b+1$. It is easy to construct a graph H with the desired property. Suppose that B_1 is of order at least 6. By induction, there exist a closed spanning walk W_1 of B_1 in which for every edge xy of B_1, x, y appear as its consecutive vertices at most twice and a connected graph H_1 with an edge cut such that $|W_1| \leq h(H_1)$, where B_1 and H_1 have the same degree sequence. We now construct a closed spanning walk W of B according to the following properties of W_1.

1. If u_r, v_r appear exactly once as consecutive vertices on W_1, then we obtain a closed spanning walk W of B from W_1 by replacing u_r, v_r by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, v_r$. Thus $|W| = |W_1| - 1 + 2r + 2 = |W_1| + 2r + 1$.

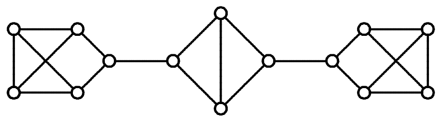
2. If u_r, v_r appear exactly twice as consecutive vertices on W_1, then we obtain a closed spanning walk W of B from W_1 by replacing u_r, v_r by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, v_r$ for the first pair and replacing u_r, v_r by $u_r, u_{r-1}, v_{r-1}, v_r$ for the second pair. Thus $|W| = |W_1| - 1 + 2r + 2 - 1 + 3 = |W_1| + 2r + 3$.

3. If u_r, v_r, u_r appear as consecutive vertices on W_1, then we obtain a closed spanning walk W of B from W_1 by replacing u_r, v_r, u_r by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, v_r, v_{r-1}, u_{r-1}, u_r$. Thus $|W| = |W_1| - 1 + 2r + 2 - 1 + 3 = |W_1| + 2r + 3$.

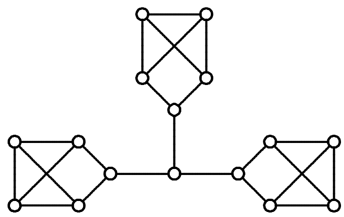
4. If u_r, v_r do not appear as consecutive vertices on W_1, then we obtain a closed spanning walk W of B from W_1 by replacing u_r by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, u_{r-1}, u_r$. Thus $|W| = |W_1| + 2r + 2 + 1 = |W_1| + 2r + 3$.

Thus W is a closed spanning walk of B and $|W| \leq |W_1| + 2r + 3$ and for every edge $e = xy \in E(B)$, x, y appear as consecutive vertices in W at most twice. On the other hand, let B' be a graph with $V(B') = V(B) - V(B_1)$ and $E(B') = E(B) - E(B_1)$. Since H_1 contains a cut edge $e = pq$, it follows that a graph H, where $V(H) = V(B') \cup V(H_1)$ and $E(H) = E(B') \cup E(H_1 - e) \cup \{u_{r-1}p, v_{r-1}q\}$, contains a cut edge. Thus $h(H) = h(B') + h(H_1) - 2 + 4 = h(H_1) + 2r + 1 + 2 = h(H_1) + 2r + 3 \geq |W|$. The proof is complete. □

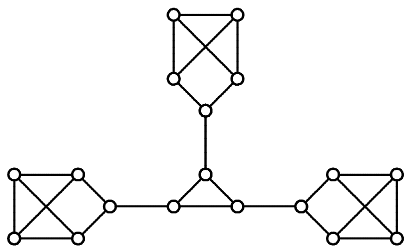
Let $G \in \mathcal{CR}_1(3^n)$ such that $n \geq 14$ and $h(G) = \max(h, 3^n)$. Then, by Lemmas 3.2.6 and 3.2.7, there exists a leaf block of G that is isomorphic to K_4^+, and by Theorem 3.2.8, all nontrivial blocks of G are $K_{2,3}$ or Hamiltonian. The following lemma is easily obtained.



G_{14}



G_{16}



G_{18}

Figure 20 Graphs G_{14}, G_{16} and G_{18}.

Lemma 3.2.9 $\max(h, 3^{14}) = 18$, $\max(h, 3^{16}) = 21$ *and* $\max(h, 3^{18}) = 24$.

Proof. We first construct graphs G_{14}, G_{16} and G_{18} with $h(G_{14}) = 18$, $h(G_{16}) = 21$ and $h(G_{18}) = 24$. Let $G \in \mathcal{CR}_1(3^{14})$ such that $h(G) = \max(h, 3^{14})$ and G contains K_4^+ as a leaf block. Thus $G = \langle K_4^+, G_1 \rangle$, where G_1 is a connected graph of order 9 containing 8 vertices of degree 3 and a vertex of degree 2. If G_1 is 2-connected, then G_1 is Hamiltonian and $h(G) = 16 < 18 = h(G_{14})$. Thus G_1 contains a cut edge. Therefore $G \cong G_{14}$. Clearly, $\max(h, 3^{14}) = 18$. Let $G \in \mathcal{CR}_1(3^{16})$ and $h(G) = \max(h, 3^{16})$. If G has only two trivial blocks, then $h(G) \leq 16 + 4 = 20 <$

$21 = h(G_{16})$. Thus G must have at least three trivial blocks. Since the order of G is 16 and G has at least two leaf block of order 5, $G \cong G_{16}$. Let $G \in \mathcal{CR}_1(3^n)$ and $h(G) = \max(h, 3^n)$. If G contains a nontrivial block of order 3, then the block is a pure triangle. We can form a new graph G' of order $n-2$ by replacing the triangle by a new vertex and joining this new vertex to the former neighbors of the triangle. Then $h(G) = h(G') + 3$. Let $G \in \mathcal{CR}_1(3^{18})$ and $h(G) = \max(h, 3^{18})$. Thus if G has a block of order 3, then $G \cong G_{18}$ and $h(G) = 24$. If G does not have a nontrivial block of order 3, then $G \cong \langle 2K_4^+, 2K_4^- \rangle$. Thus $\max(h, 3^{18}) = 24$. □

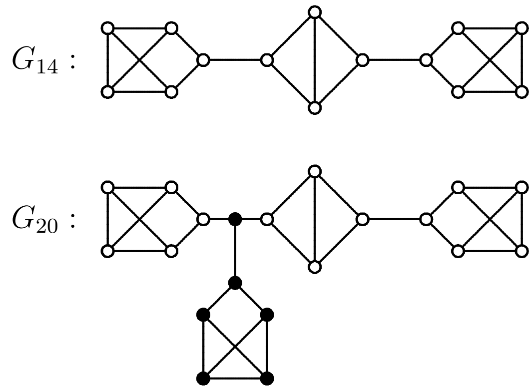


Figure 21 The Construction of the Graph G_{20}.

Let $G \in \mathcal{CR}_1(3^n)$ and xy be a cut edge of G. A graph $G(K_4(v))$ is a graph obtained from G and $K_4(v)$ by deleting the edge xy and adding the edges xz, zy, zv, where z is a new vertex. By Theorem 2.1.1, it follows that $h(G(K_4(v)) = h(G) + 9$. Note that the graph $G(K_4(v)) \in \mathcal{CR}_1(3^{n+6})$. Let $G_{14} = \langle 2K_4^+, K_4^- \rangle, G_{16} = \langle 3K_4^+ \rangle$ and $G_{18} = \langle 3K_4^+, K_3 \rangle$. Thus G_{14}, G_{16} and G_{18} are connected cubic graphs of order 14, 16, and 18, respectively. Then $h(G_{14}) = 18, h(G_{16}) = 21$ and $h(G_{18}) = 24$. Note that each of the graphs G_{14}, G_{16} and G_{18} contains a cut edge. Thus for an integer $n_i = 14 + 2i$, $i \geq 3$, a graph $G_{n_i} = G_{n_{i-3}}(K_4(v)) \in \mathcal{CR}_1(3^{n_i})$ and $h(G_{n_i}) = 18 + 3i$. We construct the graph G_{20} by the graph G_{14} as shown in Figure 21. Thus $\max(h, 3^{14+2i}) \geq 18 + 3i$, for all non-negative integer i. We show in the next theorem that the graph G_{n_i} satisfies $h(G_{n_i}) = \max(h, 3^{n_i})$.

Theorem 3.2.10 *Let i be a non-negative integer. Then* $\max(h, 3^{14+2i}) = 18 + 3i$.

Proof. We have already mentioned that $\max(h, 3^{14+2i}) \geq 18 + 3i$, for all non-negative integer i. We have also obtained $\max(h, 3^{14+2i}) = 18 + 3i$, for $i = 0, 1, 2$. Suppose that for $i \geq 3$, and $G \in \mathcal{CR}_1(3^{14+2i})$ with $h(G) = \max(h, 3^{14+2i})$. Let B be a nontrivial block of G of order b which is not a leaf block and B is of minimum order. The following five cases are considered.

Case 1. If $b = 3$, then B is a pure triangle. Let G' be a graph obtained from G by identifying B to a new vertex v, matching v to the former neighbors of B (see Figure 22(a)). Clearly $G' \in \mathcal{CR}_1(3^{14+2(i-1)})$ and by induction, $h(G) = h(G') + 3 \leq 18 + 3(i - 1) + 3 = 18 + 3i$.

Case 2. If $b = 4$ and the induced subgraph of B in G is a 4-cycle, then we can replace B by a graph in Figure 22(b) and the result follows by Case 1.

Case 3. If $b = 4$ and the induced subgraph of B in G is a K_4^-, then let G' be a graph obtained from G by deleting the K_4^- and connecting the two neighbors of K_4^- (see Figure 22(c)). Then $G' \in \mathcal{CR}_1(3^{14+2(i-2)})$ and by induction, $h(G) = h(G')+6 \leq 18 + 3(i - 2) + 6 = 18 + 3i$.

Case 4. If $b = 5$, then B is one of the graphs shown in Figure 22((d), (e), (f)) and the result follows by Case 1, Case 2 or Case 3.

Case 5. If $b \geq 6$, then, by Theorem 3.2.8, B is Hamiltonian. By an arrangement of blocks of G, we may assume that B is adjacent to a leaf block K_4^+. Let v be a vertex of degree 2 of B such that v is adjacent to the vertex of degree 2 of K_4^+. Since $b \geq 6$, we may assume that the neighbors x, y of v in B are not adjacent. Let G' be the graph obtained from G by removing $V(K_4^+) \cup \{v\}$ and adding an edge xy. Thus $G' \in \mathcal{CR}_1(3^{14+2(i-3)})$. Since $h(G) = h(G') + 1 + 7 = h(G') + 8$, by induction, $h(G) = h(G') + 8 \leq 18 + 3(i - 3) + 8 < 18 + 3i$. This completes the proof. $\square$

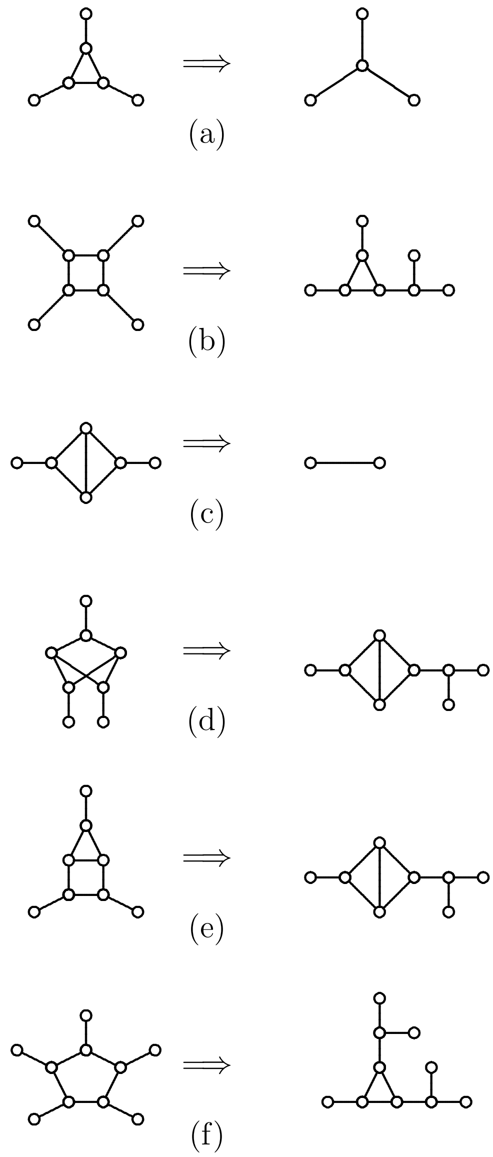


Figure 22 Parts of Block B of Graph G.

Recall that the range of the Hamiltonian numbers of connected cubic graphs of order n is $h(3^n) = \{h(G) : G \in \mathcal{CR}(3^n)\}$. It is easy to obtain $h(3^n)$ for small values of n. In fact, $h(3^n) = \{n\}$ if and only if $n = 4, 6, 8$ and $h(3^n) = \{n, n+1, n+2\}$ if and only if $n = 10, 12$. By using Theorems 2.1.1, 3.2.10 and Lemma 3.2.9, it is not difficult to show that $h(3^{14}) = \{14, 15, 16, 18\}$, $h(3^{16}) = \{16, 17, 18, 19, 20, 21\}$, $h(3^{18}) = \{18, 19, 20, 21, 22, 23, 24\}$, and $h(3^{20}) = \{20, 21, 22, 23, 24, 25, 26, 27\}$. Observe that for even integers n, $4 \le n \le 20$, $h(3^n)$ completely covers all integers from $\min(h, 3^n)$ to $\max(h, 3^n)$, except $n = 14$.

We proved in Theorem 3.1.5 that for an even integer $n \geq 10$, there exists an almost Hamiltonian cubic graph of order n. Furthermore, we proved that an almost Hamiltonian cubic graph is 2-connected. Thus for an even integer $n \geq 10$, $\{n, n+1\} \subseteq h(3^n)$.

Let n be an even integer and $n \geq 10$ and H be a Hamiltonian cubic graph of order $n-6$. Then for $G = \langle K_4(u), H(v)\rangle$, $h(G) = n+2$. Thus for an even integer $n \geq 10$, $\{n, n+1, n+2\} \subseteq h(3^n)$.

Let n be an even integer where $n \geq 16$. Consider the following elementary facts.

1. The graph $G = \langle P(u), K(v)\rangle$ satisfies $h(G) = n+3$, where P is the Petersen graph of order 10 and $K(v)$ is a Hamiltonian graph of order $n-11$ obtained from a Hamiltonian cubic graph K of order $n-12$. Thus $\{n, n+1, n+2, n+3\} \subseteq h(3^n)$.

2. The graph $G = \langle 2K_4^+, K-e\rangle$ satisfies $h(G) = n+4$, where K is a Hamiltonian cubic graph of order $n-10$ and $e \in E(K)$. Thus $\{n, n+1, n+2, n+3, n+4\} \subseteq h(3^n)$.

3. The graph $G = \langle 2K_4^+, K(v)\rangle$ satisfies $h(G) = n+5$, where $K(v)$ is a Hamiltonian graph of order $n-11$ obtained from a Hamiltonian cubic graph K of order $n-12$. Thus $\{n, n+1, n+2, n+3, n+4, n+5\} \subseteq h(3^n)$.

4. Since $\max(h, 3^{14+2i}) = 18+3i$, there exists a j-Hamiltonian graph in $\mathcal{CR}(3^{14+2i})$ the maximum value of $j = 4+i$, as constructed just before Theorem 3.2.10.

5. If $j = 4+i$, as constructed just before Theorem 3.2.10 then there exists $G \in \mathcal{CR}_1(3^{14+2i})$ such that G is a j-Hamiltonian graph in which all nontrivial blocks are Hamiltonian.

6. If $6 \leq j \leq 4+i$ and $G \in \mathcal{CR}_1(3^{14+2i})$ such that G is a j-Hamiltonian graph containing a Hamiltonian leaf block B and $v \in V(B)$, then $G*v \in \mathcal{CR}_1(3^{14+2(i+1)})$ and $G*v$ is a j-Hamiltonian graph with a Hamiltonian leaf block.

We have the following lemma.

Lemma 3.2.11 *If $n \geq 10$ and $\mathcal{CR}(3^n)$ contains a j-Hamiltonian graph, then $\mathcal{CR}(3^{n+2})$ contains a j-Hamiltonian graph.* $\square$

By Lemma 3.2.11, if $h(3^n)+2 = \{h(G)+2 : G \in \mathcal{CR}(3^n)\}$, then $h(3^n)+2 \subseteq h(3^{n+2})$. Thus by Theorem 3.2.10, we have the following theorem.

Theorem 3.2.12 *For an even integer $n \geq 16$ and $n \neq 14$. There exists an integer b such that $h(3^n) = \{k \in \mathbb{Z} : n \leq k \leq b\}$. Moreover, an explicit formula for the integer b is as follows : If $n = 14 + 2i$ and $i \geq 1$, then $b = 18 + 3i$.* $\square$

For $n \leq 14$, we know already that

1. $b = n$ if and only if $n = 4, 6, 8$.
2. $b = n + 2$ if and only if $n = 10, 12$.

3.3 The Hamiltonian Number of Some Classes of Cubic Graphs

Let G be a connected graph of order n. The *Hamiltonian coefficient* of G, denoted by $hc(G)$, is defined as $hc(G) = \frac{h(G)}{n}$. It has been shown in [2] that for every graph G of order n, $hc(G) \leq \frac{2n-2}{n} < 2$ and $hc(G) = \frac{2n-2}{n}$ if and only if G is a tree.

Let $\mathcal{CR}(3^n)$ be the class of connected cubic graphs of order n. By putting $h(3^n) = \{h(G) : G \in \mathcal{CR}(3^n)\}$, we obtained in Lemma 3.2.4 that if G is a 2-connected cubic graph of order $n \geq 10$ and $h(G) \geq n + 2$, then there exists a connected cubic graph G' of order n containing a cut edge such that $h(G) \leq h(G')$. We obtained in section 3.2 the results concerning the Hamiltonian number in the class of connected cubic graph. It should be noted that a cubic graph G_i of order $14 + 2i$ with $h(G_i) = 18 + 3i$ is a graph containing as many cut edges as possible. Furthermore, $\frac{h(G_i)}{|n(G_i)|} = \frac{18+3i}{14+2i} < \frac{3}{2}$ and

$$\lim_{i\to\infty} hc(G_i) = \frac{3}{2}.$$

The problem of finding the maximum value of $hc(G)$ in the class of 2-connected cubic graphs of order n is not easy. We introduce three classes of 2-connected cubic graphs with small circumference and hence large Hamiltonian number.

Let G be a connected graph of order n and $W : w_1, w_2, w_3, \ldots, w_\ell, w_1$ be a Hamiltonian walk of G. Let $W_1(G)$ be the set of vertices $v \in V(G)$ such that v appears in $w_1, w_2, w_3, \ldots, w_\ell$ exactly once and $W_2(G) = V(G) - W_1(G)$. Thus G is Hamiltonian if and only if $W_2(G) = \emptyset$. Furthermore, $h(G) \geq n + |W_2(G)|$. Let $e = uv \in E(G)$. Then e is said to appear in W if u and v appear as consecutive vertices on W.

Lemma 3.3.1 *Let G be a connected cubic graph of order n and W be a Hamiltonian walk of G. If X is the set of $v \in V(G)$ such that all edges incident with v appear in W, then $X \subseteq W_2(G)$.*

Proof. Let $W : w_1, w_2, \ldots, w_\ell, w_1$ be a Hamiltonian walk of a connected cubic graph G of order n. Since for each i, $1 \leq i \leq \ell$, w_i can have at most two distinct neighbors along W, it follows that if xv, yv, zv appear in W, then v must appear at least twice on W. Thus $v \in W_2(G)$. □

Let G be a connected graph and $uv \in E(G)$. A *subdivision* of the edge uv is the operation of replacing uv with a path u, w, v through a new vertex w.

Lemma 3.3.2 *Let G be a connected graph. If G^+ is a graph obtained from G by subdividing an edge uv of G, then $h(G^+) \geq h(G) + 1$.*

Proof. Let $W' : w_1, w_2, \ldots, w_\ell, w_1$ be a Hamiltonian walk of G^+. Let w be the new vertex of a subdivision of the edge uv of G. It is clear that there is a unique i in which $w_i = w$ and we can assume that $w_1 = w$. If $w_2 \neq w_\ell$, then $W : w_2, w_3, \ldots, w_\ell, w_2$ is a closed spanning walk of G. If $w_2 = w_\ell$, then $W : w_2, w_3, \ldots, w_\ell$ is a closed spanning walk of G. Thus $h(G) \leq h(G^+) - 1$. □

These results and notation will be used in subsections 3.3.2 and 3.3.3.

3.3.1 Petersen Graphs

Let $P = P(5, 2)$ be the Petersen graph of order 10. It is well known (for example see [14]) that P has the following properties.

1. It is a vertex transitive and edge transitive graph.
2. It is not Hamiltonian and $h(P) = 11$.
3. Let $u, v \in V(P)$ and $u \neq v$. Let $W : u = u_1, u_2, \ldots, u_t = v$ be a spanning walk in P. Then $t = 10$ if $uv \notin E(P)$, otherwise $t = 11$.

Let $P(k)$ be a cubic graph with $10k$ vertices formed by k pairwise disjoint copies of $P - e$ by adding k edges to link them in a ring as shown in Figure 23. More precisely, let $P_1, P_2, \ldots, P_k$ be k pairwise vertex disjoint copies of P and $u_i, v_i \in V(P_i)$ and $u_iv_i \in E(P_i)$. Let $P(k)$ be the graph with $V(P(k)) = \bigcup_{j=1}^{k} V(P_j)$ and $E(P(k)) = (\bigcup_{j=1}^{k} E(P_j - u_jv_j)) + \{u_1v_2, u_2v_3, \ldots, u_{k-1}v_k, u_kv_1\}$. Then $P(k)$ is a 2-connected cubic graph of order $10k$.

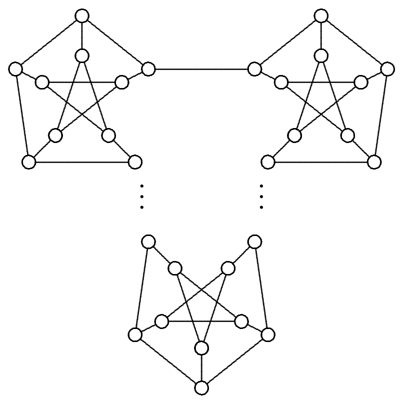


Figure 23 Graph $P(k)$.

Theorem 3.3.3 *Let k be an integer with $k \geq 2$. Then $h(P(k)) = 11k$.*

Proof. Let $P(k)$ be the graph as described above. It is easy to obtain a closed spanning walk of length $11k$ of $P(k)$. Thus $h(P(k)) \leq 11k$. Let

$$W : w_1, w_2, \ldots, w_t, w_{t+1} = w_1$$

be a Hamiltonian walk of $P(k)$. Let $w_{i_1}, w_{i_2}, \ldots, w_{i_\ell}$ be the subsequence of W consisting of those vertices of P_i. Thus $1 \leq i_1 < i_2 < \ldots < i_\ell \leq t$ and $\ell \geq 10$. Suppose that there exists an integer $j = 1, 2, \ldots, \ell$ such that $i_{j+1} - i_j > 1$. Then $w_{i_1}, w_{i_j}, w_{i_{j+1}}, w_{i_\ell} \in \{u_i, v_i\}$. If $\ell = 10$, then $w_{i_1}, w_{i_2}, \ldots, w_{i_\ell}$ are all distinct. Thus $\ell \geq 12$, which is a contradiction. Therefore, for all $j = 1, 2, \ldots, \ell$, $i_{j+1} - i_j = 1$ and in this case the sequence $w_{i_1}, w_{i_2}, \ldots, w_{i_\ell}$ forms a u_i, v_i spanning path of P_i. Thus $\ell \geq 11$. This completes the proof. $\square$

The result of Theorem 3.3.3 implies that $hc(P(k)) = 1.1$ for all $k \geq 1$.

3.3.2 Cubic Graphs with Small Circumference

We introduce another class of 2-connected cubic graphs which can be defined in the following way. For a positive integer $k \geq 2$, let $H(k)$ be a cubic graph as shown in Figure 24.

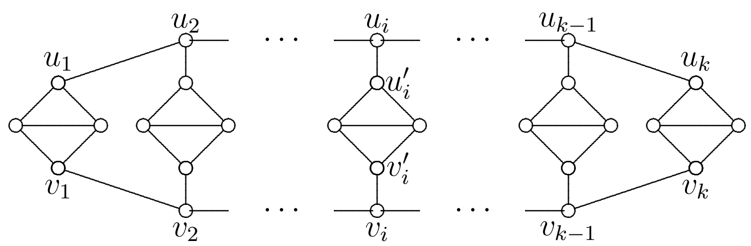


Figure 24 Graph $H(k)$.

The graph $H(k)$ has the following properties.

1. $H(2)$ is Hamiltonian.
2. It was shown in [22] that $H(k)$ is of order $6k - 4$ having circumference $2k + 4$. Thus $H(k)$ is not Hamiltonian if $k \geq 3$.

We proved in Theorem 3.1.7 that a cubic graph G of order $n \geq 10$ with $h(G) = n+1$ if and only if G is 2-connected having circumference $n-1$. Furthermore we proved that for every even integer $n \geq 10$, there exists a 2-connected cubic graph

G of order n with $h(G) = n + 1$. Since $H(3)$ is 2-connected cubic graph of order 14 with circumference 10, it follows that $h(H(3)) \geq 16$. It is easy to produce a closed spanning walk of $H(3)$ of length 16. Thus $h(H(3)) = 16$. Similarly, the graph $H(4)$ is of order 20 with $h(H(4)) = 22$. We now suppose that $k \geq 5$. Let $H(k)$ with vertices $u_1, u_2, \ldots, u_k, v_k, v_{k-1}, \ldots, v_2, v_1$ on its circumference as shown in Figure 24. Thus $H(k) - \{u_i u_{i+1}, v_i v_{i+1}\}$ is a disconnected graph containing two components G_1 and G_2, where G_1 contains u_i and G_2 contains u_{i+1}. Summarizing we have the following results.

1. If $i = 1$, then $G_1 = K_4 - e$, for some edge e of K_4, and G_2 is a graph obtained from $H(k-1)$ with two subdivisions of edges. Thus $h(H(k) - u_1 u_2) = h(G_1) + h(G_2) + 2 \geq 4 + h(H(k-1)) + 2 + 2 = h(H(k-1)) + 8$.

2. If $i \geq 2$ and $k - i \geq 2$, then G_1 is a graph obtained from $H(i)$ with two subdivisions of edges and G_2 is obtained from $H(k-i)$ with two subdivisions of edges. Thus $h(H(k) - u_i u_{i+1}) \geq h(H(i)) + 2 + h(H(k-i)) + 2 + 2 = h(H(i)) + h(H(k-i)) + 6$.

3. For $i \geq 2$ we have that u_i is adjacent to u_{i-1} and u_{i+1}. Put u_i' as the third vertex that is adjacent to u_i. Similarly, v_i' is the third vertex that is adjacent to v_i. Thus $h(H(k) - u_i u_i') \geq 4 + h(H(k-1)) + 2 + 2 = h(H(k-1)) + 8$.

4. If k is even and $k \geq 4$, then, by Theorem 2.1.1, $h(H(k)) \leq h(H(k) - \{u_2 u_3, u_4 u_5, \ldots, u_{k-2} u_{k-1}\}) = 6k - 4 + 2(k-2)/2 = 7k - 6$.

5. If k is odd and $k \geq 5$, then, by Theorem 2.1.1, $h(H(k)) \leq h(H(k) - \{u_2 u_3, u_4 u_5, \ldots, u_{k-1} u_k\}) = 6k - 4 + 2(k-1)/2 = 7k - 5$.

6. Let W be a Hamiltonian walk of $H(k)$. If $i \geq 2$ and u_{i-1}, u_i, u_{i+1} and u_i' appear in W, then, by Lemma 3.3.2, u_i must appear at least twice on W. Similarly for v_i. Thus if each of vertices $u_2, u_3, \ldots, u_{k-1}, v_2, v_3, \ldots, v_{k-1}$ appears at least twice in W, then $h(H(k)) \geq 6k - 4 + 2(k-2) = 8k - 8$.

7. If $k \geq 4$, then there exists i $(1 \leq i \leq k-1)$ such that $h(H(k)) = h(H(k) - e)$ where $e \in \{u_i u_{i+1}, u_i u_i'\}$.

The following theorem can be obtained from above observation.

Theorem 3.3.4 *Let k be an integer with $k \geq 2$. Then*

$$h(H(k)) = \begin{cases} 7k-5 & if\ k\ is\ odd, \\ 7k-6 & if\ k\ is\ even. \end{cases}$$

□

The result of Theorem 3.3.4 implies that $hc(H(k)) = \frac{7k-5}{6k-4} < \frac{7}{6}$ for odd integers $k \geq 3$, $hc(H(k)) = \frac{7k-6}{6k-4} < \frac{7}{6}$ for even integers $k \geq 2$, and

$$\lim_{k\to\infty} hc(H(k)) = \frac{7}{6}.$$

3.3.3 Cubic Graphs with Even Smaller Circumference

The graph $H(k)$ is a 2-connected cubic graph of order $6k-4$ with circumference $2k+6$. Thus the circumference is about one third of its order. We introduce in this subsection a class of 2-connected cubic graphs $G(k)$, $k \geq 2$ shown in Figure 25. The graph $G(k)$ has order $12k-4$ with circumference $2k+8$ which is about one sixth of the order [22].

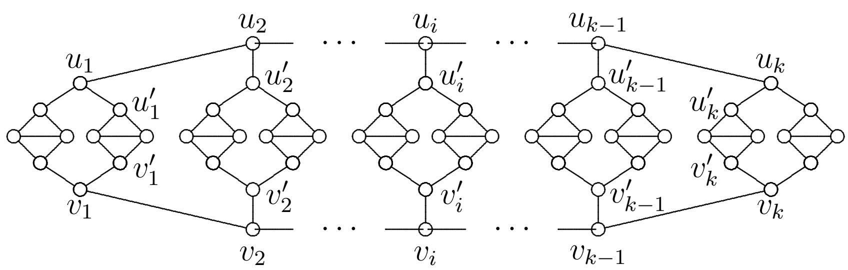


Figure 25 Graph $G(k)$.

In order to obtain $h(G(k))$ we first observe the following elementary facts.

1. The graph $G(2)$ is of order 20 and of circumference 12. It is not Hamiltonian as $h(G(2)) \geq 22$. It is easy to produce a closed spanning walk of $G(2)$ of length 22 and hence $h(G(2)) = 22$.

2. By Theorem 2.1.1, $h(G(k)) \leq h(G(k) - \{u_1u_1', u_2u_2', \ldots, u_ku_k'\}) = 12k - 4 + 2k = 14k - 4$.

Let $G^*(k)$ be a graph obtained from $G(k)$ by subdividing the edge u_1u_2 and the edge v_1v_2. Put x and y to be the inserted vertices in u_1u_2 and v_1v_2, respectively. It is clear that $h(G(k))+2 \leq h(G^*(k)) \leq h(G(k))+4$. We have the following result.

Lemma 3.3.5 *$h(G^*(2)) = 26$ and $h(G(3)) = 38$.*

Proof. Let $W : w_1, w_2, \ldots, w_\ell, w_1$ be a Hamiltonian walk of $G^*(2)$. If $u_1, u_2, v_1, v_2 \in W_2(G^*(2))$, then $h(G^*(2)) \geq 22 + 4 = 26$. Suppose, without loss of generality, that $u_1 \in W_1(G^*(2))$. Then $h(G^*(2)) = h(G^*(2) - u_1x)$ or $h(G^*(2)) = h(G^*(2) - u_1u_1')$. By applying Theorem 2.1.1 in both cases, we can conclude that $h(G^*(2)) = 26$.

Observe that the graph $G(3)$ has order 32. Let $W : w_1, w_2, \ldots, w_\ell, w_1$ be a Hamiltonian walk of $G(3)$. If $u_1, u_2, u_3, v_1, v_2, v_3 \in W_2(G(3))$, then $h(G(3)) \geq 32 + 6 = 38$. Suppose, without loss of generality, that $h(G(3)) = h(G(3) - u_2u_2')$, $h(G(3)) = h(G(3) - u_1u_2)$ or $h(G(3)) = h(G(3) - u_1u_1')$. By using the result of Theorem 2.1.1 and $h(G^*(2)) = 26$, $h(G(3) - u_2u_2') = 26 + 10 + 2 = 38$ and $h(G(3) - u_1u_2) = 26+10+2 = 38$. In order to calculate $h(G(3)-u_1u_1')$, we may assume further that every Hamiltonian walk of $G(3)-u_1u_1'$ contains $u_1u_2, u_2u_3, u_2u_2', v_1v_2, v_2v_3, v_2v_2'$. Thus, by Lemma 3.3.1, $u_2, v_2 \in W_2(G(3))$. Since $G(3) - u_1u_1'$ contains three blocks one of which is of order 2 and the other of order 4 having v_1' as the common vertex. Thus $v_1, v_1' \in W_2(G(3))$. There are also three possibilities for u_3, u_3', v_3, v_3' for W, namely $u_3, v_3 \in W_2(G(3))$, $u_3, u_3' \in W_2(G(3))$ or $v_3, v_3' \in W_2(G(3))$. Therefore $h(G(3)) = 38$, as required. □

Theorem 3.3.6 *Let k be an integer with $k \geq 3$. Then $h(G(k)) = 14k - 4$.*

Proof. We will proceed by induction on k. The result holds for $k = 3$. Suppose that the result holds for $k - 1 \geq 3$. Let $G(k)$ be the graph as shown in Figure 11 and W be a Hamiltonian walk of $G(k)$. Suppose that for each $i = 1, 2, \ldots, k$, $u_i, v_i \in W_2(G(k))$. Then $h(G(k)) = |W| \geq 12k-4+2k = 14k-4$. Suppose, without loss of generality, that there exists i such that $u_i \notin W_2(G(k))$. If $2 \leq i \leq k-2$, then

$h(G(k)) = h(G(k) - u_iu_i')$ or $h(G(k)) = h(G(k) - u_iu_{i+1})$. Since $G - u_iu_i'$ consists of three blocks, one of which is isomorphic to $G^*(k-1)$. Thus, by Theorem 2.1.1 and induction, $h(G(k)) = h(G(k) - u_iu_i') = h(G^*(k-1)) + 10 + 2 \geq 14(k-1) - 4 + 2 + 10 + 2 = 14k - 4$. The graph $G(k) - u_iu_{i+1}$ consists of three blocks, one of which is $G^*(i)$ and the other is $G^*(k-i)$. Thus, by Lemma 3.3.5, Theorem 2.1.1 and induction, $h(G(k)) = h(G^*(i)) + h(G^*(k-i)) + 2 \geq 14i - 4 + 2 + 14(k-i) - 4 + 2 + 2 = 14k - 2 > 14k - 4$. If $u_1 \notin W_2(G(k))$, then $h(G(k)) = h(G(k) - u_1u_2)$ or $h(G(k)) = h(G(k) - u_1u_1')$. If $h(G(k)) = h(G(k) - u_1u_2)$, then $h(G(k)) = h(G(k) - u_1u_2) = 10 + h(G^*(k-1)) + 2 \geq 10 + 14(k-1) - 4 + 2 + 2 = 14k - 4$. If $h(G(k)) = h(G(k) - u_1u_1')$, then, by above argument, we may assume that for each $1 \leq i \leq k-1$, $u_i, v_i \in W_2(G(k))$. Since the graph $G(k) - u_1u_1'$ consists of three blocks and by Theorem 2.1.1, it follows that $v_1, v_1' \in W_2(G(k))$. It can be shown that $u_k, v_k \in W_2(G(k))$ or $u_k, u_k' \in W_2(G(k))$ or $v_k, v_k' \in W_2(G(k))$. Therefore, $h(G(k)) = 14k - 4$. □

The result of Theorem 3.3.6 implies that $hc(G(k)) = \frac{14k-4}{12k-4} > \frac{7}{6}$ for integers $k \geq 2$, and

$$\lim_{k \to \infty} hc(G(k)) = \frac{7}{6}.$$

3.4 The Hamiltonian Number of Graphs with Prescribed Connectivity

One of the interesting properties of 2-connected graphs is that any two vertices of such graphs lie on a common cycle. There is a generalization of this fact to k-connected graphs by Dirac [10] as we state in the following theorem.

Theorem 3.4.1 *Let G be a k-connected graph, $k \geq 2$. Then every k vertices of G lie on a common cycle of G.* □

Our next result involves the independence number and the connectivity of a graph. This result is due to Chvátal and Erdős [9].

Theorem 3.4.2 *Let G be a graph with at least three vertices. If $\kappa(G) \geq \beta(G)$,* *then G is Hamiltonian.* □

Let $\mathcal{G}(n)$ be the set of all connected graphs of order n. Then $\mathcal{G}(n)$ can be partitioned according to the connectivity. For integers n and k with $1 \le k \le n-1$, we put

$$\mathcal{G}(n, \kappa = k) = \{G \in \mathcal{G}(n) : \kappa(G) = k\} \text{ and } h(n,k) = \{h(G) : G \in \mathcal{G}(n, \kappa = k)\}.$$

Furthermore, we denote by

$$\min(h; n, k) := \min\{h(G) : G \in \mathcal{G}(n, k)\} \text{ and}$$

$$\max(h; n, k) := \max\{h(G) : G \in \mathcal{G}(n, \kappa = k)\}.$$

We prove in this section that for any pair of integers n, k with $1 \le k \le n-1$, there exist positive integers $a := \min(h; n, k)$ and $b := \max(h; n, k)$ such that $h(n, k) = \{x \in \mathbb{Z} : a \le x \le b\}$. Moreover, the values of $\min(h; n, k)$ and $\max(h; n, k)$ are obtained in all situations.

We first consider when $k = 1$ and $n \ge 3$. Since a Hamiltonian graph of order $n \ge 3$ is 2-connected, it follows that $\min(h; n, 1) \ge n+1$. Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_n v_i : i = 1, 2, \ldots, n-1\}$. Thus G is a star of order n with center at v_n, $G \in \mathcal{G}(n, \kappa = 1)$ and, by Theorem 2.1.1, $h(G) = 2n-2$. Put $G_0 = G, G_1 = G_0 + v_1v_2, G_2 = G_1 + v_2v_3, \ldots, G_{n-3} = G_{n-4} + v_{n-3}v_{n-2}$ (see Figure 26). Thus $G_i \in \mathcal{G}(n, \kappa = 1)$, for each $i = 1, 2, \ldots, n-3$, and, by Theorem 2.1.1, $h(G_i) = 2n - 2 - i$. Therefore, $h(n, 1) = \{x \in \mathbb{Z} : n+1 \le x \le 2n-2\}$. Thus we have proved the following theorem.

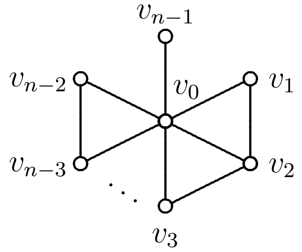


Figure 26 The Graph G_{n-3}.

Theorem 3.4.3 *Let n be a positive integer with $n \geq 3$. Then $h(n, 1) = \{x \in \mathbb{Z} : n + 1 \leq x \leq 2n - 2\}$.* □

For given integers n and k with $2 \leq k \leq n - 1$, a graph G obtained from K_{n-1} by joining a new vertex v to k vertices of K_{n-1} satisfies $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = n$. Thus $\min(h; n, k) = n$.

Lemma 3.4.4 *Let $G = (V, E)$ be a connected graph of order n and $E_1 = \{e_1, e_2, \ldots, e_t\} \subseteq E(G)$. If $\langle E_1 \rangle$ contains no cycle, then there exists a spanning tree T of G such that $E_1 \subseteq E(T)$.*

Proof. We will proceed by induction on t. Suppose that $t = 1$. Let T_1 be a spanning tree of G and $e_1 \notin E(T_1)$. Then $T_1 + e_1$ contains a cycle. Thus there exists $f \in E(T_1)$ such that $T_1 + e - f$ is a spanning tree of G containing e_1. Therefore the result holds for $t = 1$. We now suppose that $t \geq 2$ and the result holds for the graph $E_1 - \{e_t\}$. That is, there exists a spanning tree T_1 of G such that $E_1 - \{e_t\} \subseteq E(T_1)$. Thus $T_1 + e_t$ contains a unique cycle C. Since $\langle E_1 \rangle$ is a subgraph of $T_1 + e_t$ and $\langle E_1 \rangle$ does not contain a cycle, C contains an edge f such that $f \notin E_1$. Therefore $T = T_1 + e_t - f$ is a spanning tree of G such that $E_1 \subseteq E(T)$ as required. □

As an application we obtain an upper bound for the Hamiltonian number of a connected graph containing a cycle.

Lemma 3.4.5 *Let G be a connected graph of order n. If G contains a cycle of order k, then $h(G) \leq 2n - k$.*

Proof. We first note that if G is a connected graph and $e \in E(G)$ such that $G - e$ is connected, then $h(G) \leq h(G - e)$. Let C be a cycle in G of order k and $e \in E(C)$. By Lemma 3.3.2, let T be a spanning tree of G containing $E(C - e)$. Thus $T + e$ consists of $n - k + 1$ blocks $B_1, B_2, \ldots B_{n-k+1}$ such that B_1 is a cycle of order k and the rest are blocks of order two. Thus, by Theorem 2.1.1, $h(G) \leq h(T + e) = k + 2(n - k) = 2n - k$.

□

The following lemma provides a lower bound for the Hamiltonian number of a graph in term of the independence number of the graph.

Lemma 3.4.6 *Let G be a connected graph of order n. Then $h(G) \geq 2\beta(G)$. In particular, if G is Hamiltonian, then $\beta(G) \leq \frac{n}{2}$.*

Proof. Let $W : u_0, u_1, \ldots, u_t = u_0$ be a Hamiltonian walk of G. Let $S = \{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}$ be a maximum independent set of G such that $0 \leq i_1 < i_2 < \ldots < i_r \leq t$. Thus $r = \beta(G)$. Since S is an independent set of vertices, it follows that for $j = 1, 2, \ldots, r-1$, $i_{j+1} - i_j \geq 2$. Thus $t \geq 2r$. This completes the proof. □

Lemma 3.4.7 *Let n and k be positive integers. Then $h(n, k) = \{n\}$ if and only if $n \leq 2k$.*

Proof. Suppose that $n \leq 2k$. Let G be a k-connected graph of order n and I be a maximum independent set of vertices of G. Since G is not $(k+1)$-connected, $k \leq \delta(G)$. Thus for each $v \in I$, v has at least k neighbors in $V(G) - I$. It follows that G has at least $|I| + k$ vertices and hence $|I| + k \leq n$. Since $n \leq 2k$, $\beta(G) = |I| \leq n - k \leq k$. Thus, by Theorem 3.4.2, G is Hamiltonian. Conversely, suppose that $n > 2k$. Let G be a graph with $V(G) = I \cup K$, where $I = \{v_1, v_2, \ldots, v_{n-k}\}$ and $K = \{w_1, w_2, \ldots, w_k\}$, and $E(G) = \{w_i w_j : 1 \leq i < j \leq k\} \cup \{v_i w_j : i = 1, 2, \ldots, n-k, j = 1, 2, \ldots, k\}$. It is clear that $G \in \mathcal{G}(n, \kappa = k)$ and I is an independent set of vertices of G. Since $|I| = n-k$ and Lemma 3.4.6, $h(G) \geq 2(n-k) = n+(n-2k) > n$. Therefore $h(n, k) \neq \{n\}$. □

The result of Lemma 3.4.7 gives a characterization of $h(n, k) = \{n\}$ as $k \geq n/2$. So we may assume from now on that $k < n/2$.

A graph $G = (V, E)$ is called a *split graph* if there exists a partition $\{I, K\}$ of V such that the subgraphs $\langle I \rangle$ and $\langle K \rangle$ of G induced by I and K are empty and complete graphs, respectively. Note that if $G = (V, E)$ is a split graph, then the corresponding partition $\{I, K\}$ of V may not be unique. It is unique if we choose the corresponding partition $\{I, K\}$ of V with minimum cardinality $|K|$. Thus for a split graph $G = (V, E)$, we understand that the corresponding partition $\{I, K\}$ of V is chosen in such a way that K has minimum cardinality. We will denote such a graph by $S(I \cup K, E)$. Further, a split graph $G = S(I \cup K, E)$ is called a *complete split graph* if for every vertex $v \in I$, v is adjacent to every vertex in K. Thus if G is a complete split graph of order n, then there exists a unique pair of integers k and $n-k$ such that $|K| = k$ and $|I| = n-k$. In this particular case, we denote G by $CS(n-k, k)$. Thus $K_n = CS(1, n-1)$, for all $n \geq 2$. It is easy to see that $\kappa(CS(n-k, k)) = k$.

A split graph $G = S(I \cup K, E)$ with $|I| = |K|$ has a Hamiltonian cycle if and only if the bipartite graph $G' = G - E(\langle K \rangle)$ has a Hamiltonian cycle. It is not difficult to show that a split graph $G = S(I \cup K, E)$ with $|I| < |K|$ contains a Hamiltonian cycle if and only if the graph $\langle I \cup N_G(I) \rangle$ contains a Hamiltonian cycle. Further, G contains no Hamiltonian cycle if $|I| > |K|$.

The complete split graph $G = CS(n-k, k)$, $k \geq 2$, satisfies the conditions that $\kappa(G) = k$ and $\beta(G) = n-k$. The following result can be considered as a direct consequence of Lemma 3.4.7 and Theorem 2.1.2.

Corollary 3.4.8 *Let $G = CS(n-k, k)$ be a complete split graph of order n and $k \geq 1$. Then G has a Hamiltonian cycle if and only if $n \leq 2k$. Moreover, if $n > 2k$, then $h(G) = 2(n-k)$.* □

We are now ready to prove the following main results.

Theorem 3.4.9 *Let n and k be integers such that $k \geq 2$ and $n > 2k$. Then $\min(h; n, k) = n$ and $\max(h; n, k) = 2(n-k)$. Moreover, for any positive integer i such that $0 \leq i \leq n - 2k$, there exists $G_i \in \mathcal{G}(n, \kappa = k)$ with $h(G_i) = 2(n-k) - i$.*

Proof. We have already mentioned earlier that $\min(h; n, k) = n$ for all pairs of integers n, k such that $k \geq 2$ and $n \geq 2k$. It is clear that $CS(n-k, k) \in \mathcal{G}(n, \kappa = k)$. Since $h(CS(n-k, k)) = 2(n-k)$, $\max(h; n, k) \geq 2(n-k)$. On the other hand, let $G \in \mathcal{G}(n, \kappa = k)$. If $\beta(G) \leq k$, then, by Theorem 3.4.2, $h(G) = n < n + (n-2k) = 2(n-k)$. Now suppose that $\beta(G) > k$. Let $X = \{v_1, v_2, \ldots, v_k\}$ be a set of k

independent vertices of G. By Theorem 3.4.1, there exists a cycle C in G containing $v_1, v_2, \ldots, v_k$. Since X is an independent set, C has order at least $2k$. By Lemma 3.4.5, $h(G) \leq 2n - 2k = 2(n-k)$. Thus $\max(h; n, k) = 2(n-k)$.

Let $G = CS(n-k, k)$ such that $V(G) = I \cup K$, $I = \{v_1, v_2, \ldots, v_{n-k}\}$ and $K = \{w_1, w_2, \ldots, w_k\}$. Put $G = G_0, G_1 = G_0 + v_k v_{k+1}, G_2 = G_1 + v_{k+1} v_{k+2}, \ldots, G_{n-2k} = G_{n-2k-1} + v_{n-k-1} v_{n-k}$. Thus $\beta(G_i) = n - k - \lceil i/2 \rceil$, for all $i = 0, 1, 2, \ldots, n-2k$. Also, G_i contains a cycle of order $2k + i$, for all $i = 1, 2, \ldots, n-2k$. Thus, by Lemmas 3.4.5 and 3.4.6, we have that for all $i = 0, 1, 2, \ldots, n-2k$, $2(n-k-\lceil i/2 \rceil) \leq h(G_i) \leq 2n - 2k - i$. Further, G_{n-2k} contains a cycle of order $2k + n - 2k = n$. Thus $h(G_{n-2k}) = n$. It is clear that $2(n-k-\lceil i/2 \rceil) = 2(n-k) - i$ if i is even and $2(n-k-\lceil i/2 \rceil) = 2(n-k) - i - 1$ if i is odd, we have that $h(G_i) = 2(n-k) - i$, for all even integers i with $0 \leq i < n - 2k$. We now consider for odd integer i. Let $W : u_0, u_1, \ldots, u_{t-1}, u_t = u_0$ be a Hamiltonian walk of G_i. Then there exist $u_{i_1}, u_{i_2}, \ldots, u_{i_{n-k}}$ such that $0 \leq i_1 < i_2 < \ldots < i_{n-k} \leq t$ and $\{u_{i_1}, u_{i_2}, \ldots, u_{i_{n-k}}\} = \{v_1, v_2, \ldots, v_{n-k}\}$. Observe that $\{v_1, v_2, \ldots, v_{k-1}, v_{k+i+1}, \ldots, v_{n-k}\}$ is an independent set of $n-k-i-1$ vertices and $v_k, v_{k+1}, \ldots, v_{k+i}$ is a path of order $i+1$ of G_i. It follows that $|W| \geq 2(n-k-i-1) + i + 1 + 1 = 2(n-k) - i$. Therefore $h(G_i) = 2(n-k) - i$ as required. Thus we have $h(n, k) = \{x \in \mathbb{Z}^+ : n \leq x \leq 2(n-k)\}$. □

We have seen that for integers $n \geq 3$ and $k \geq 1$ such that $n > 2k$, the graph $CS(n-k, k)$ satisfies the following properties:

1. $CS(n-k, k)$ is not Hamiltonian and $h(CS(n-k, k)) = 2(n-k)$,

2. $CS(n-k, k) \in \mathcal{G}(n, \kappa = k)$,

3. if $G \in \mathcal{G}(n, \kappa = k)$, then $h(G) \leq h(CS(n-k, k)) = 2(n-k)$,

4. $CS(n-k, k)$ is a graph of size $\binom{k}{2} + k(n-k)$.

If $k = 1$, then a characterization of graph G of order n with $h(G) = 2(n-1)$ can be obtained by result of Theorem 2.1.1. Let $n \geq 3$ and $k \geq 2$ be integers with $n > 2k$. If $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = 2(n-k)$, then we have the following facts.

1. Since $h(G) = 2(n-k) = n+(n-2k) > n$, it follows that G is not Hamiltonian. Thus, by Theorem 3.4.2, $\beta(G) \geq k+1$.

2. If $\{v_1, v_2, \ldots, v_k\}$ is an independent set of k vertices of G, then, by Theorem 3.4.1, G contains a cycle of order at least $2k$. Since $h(G) = 2(n-k)$ and by Lemma 3.4.5, it follows that G contains a cycle of order at most $2k$. Thus G contains a cycle of order $2k$.

The following theorem is a characterization of k-connected graph of order n having Hamiltonian number $2(n-k)$.

Theorem 3.4.10 *Let $n \geq 3$ and $k \geq 2$ be integers with $n > 2k$. If $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = 2(n-k)$, then $m(G) \leq m(CS(n-k, k))$. Further, if $G \in \mathcal{G}(n, \kappa = k)$, then $h(G) = 2(n-k)$ and $m(G) = m(CS(n-k,k))$ if and only if $G \cong CS(n-k, k)$.*

Proof. Let $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = 2(n-k)$. By the fact (2) above there exists a cycle C of G of order $2k$ containing $\{v_1, v_2, \ldots, v_k\}$ and G does not contain a cycle of order more than $2k$. Without loss of generality, we may assume that $C : v_1, w_1, v_2, w_2, \ldots, v_k, w_k, v_1$. Let $X = V(G) - V(C)$. Then $|X| = n - 2k$. Since $h(G) = 2(n-k)$, $\langle X \rangle$ contains no cycle. Let K be a component of $\langle X \rangle$. Then $|N_G(V(K)) \cap V(C)| \leq k$ since otherwise G must contain a cycle of order at least $2k+1$. Suppose that K has order at least 2. If there exist two vertices of K have a common neighbor in C, then $h(G) < 2(n-k)$. Thus the average degree of all vertices of K is less than k. This is a contradiction. Thus $\langle X \rangle$ is an empty graph and for each $v \in X$ and $d(v) = k$. Let $v_{k+1} \in X$ such that $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ forms an independent set of G. Thus v_{k+1} is adjacent to $w_1, w_2, \ldots, w_k$. Further, for each $v \in X$, v is adjacent to either $v_1, v_2, \ldots, v_k$ or $w_1, w_2, \ldots, w_k$, otherwise, G must contain a cycle of order at least $2k+1$. Suppose that there exists $v \in X$ such that v is adjacent to $v_1, v_2, \ldots, v_k$. Then $\langle X \cup \{v, v_{k+1}\} \rangle$ contains a cycle of order $2k+2$. Thus for each $v \in X$, $N_G(v) = \{w_1, w_2, \ldots, w_k\}$. Therefore, $\{v_1, v_2, \ldots, v_k\} \cup X$ is an independent set of G of cardinality $n-k$ which implies that G is a subgraph of $CS(n-k, k)$. Thus, $m(G) = m(CS(n-k,k))$ if and only if $G \cong CS(n-k,k)$. $\square$

By Theorem 3.4.10, we have that the complete split graph $CS(n-k,k)$ is the only k-connected graph of order n with Hamiltonian number $2(n-k)$ and of maximum size.

CHAPTER 4

SUMMARY AND OPEN PROBLEMS

In this dissertation, we have found several significant results concerning the Hamiltonian number of graphs. More precisely, we found the range of Hamiltonian numbers in some classes of graphs, for example, the class of connected cubic graphs of order n, the class of graphs of order n with connectivity κ. We summarize our results in Section 4.1 and we will list some open problems in this direction that we have encountered in the past few years during our study for this graph parameter in Section 4.2.

4.1 Summary

This section summarizes our comprehensive work concerning interpolation and extremal results for graph parameter h. The results obtained can be divided in 4 parts. The first part deals with our work on the Hamiltonian number of some classes of cubic graphs. In particular, of the generalized Petersen graphs, motivated by a result on the Petersen graph. It is well known that the Petersen graph $P(5,2)$ is not Hamiltonian and its Hamiltonian number $h(P(5,2))$ is 11. It was shown by Alspach [1] that the generalized Petersen graph $P(k,m)$ is non-Hamiltonian if and only if $m = 2$ and $k \equiv 5 (\text{mod } 6)$. We proved in section 3.1 that $h(6k+5,2) = 12k+11$ for all non-negative integers k and for every integer $n \geq 10$ there exists a connected cubic graph G_n of order n with $h(G_n) = n+1$. Finally, we obtained a characterization of cubic graphs G of order n having $h(G) = n+1$, and $n \geq 10$: $h(G) = n+1$ if and only if G is 2-connected cubic graph with circumference $n-1$.

The problem of determining the range of Hamiltonian number in the class of connected cubic graphs is considered in section 3.2. We obtained several significant results in the class of connected cubic graphs, in particular, we proved in this section that if G is a connected cubic graph of order $n \geq 10$, then there exists a connected cubic graph G' of order n containing a cut edge and $h(G) \leq h(G')$. Therefore in order to obtain a cubic graph having large Hamiltonian number, we focused on cubic graphs containing as many cut edges as possible. With this observation, we were able to prove in this section that if G runs over the set of connected cubic graphs of order n and $n \neq 14$, then the values $h(G)$ completely cover a line segment $[a, b]$ of positive integers. We defined the range of Hamiltonian numbers, $h(3^n)$, by letting $\mathcal{CR}(3^n)$ to be the class of all connected cubic graphs of order n and

$$h(3^n) = \{h(G) : G \in \mathcal{CR}(3^n)\}.$$

Thus the range of Hamiltonian numbers $h(3^n)$ is uniquely determined by $\min(h, 3^n) = \min\{h(G) : G \in \mathcal{CR}(3^n)\}$ and $\max(h, 3^n) = \max\{h(G) : G \in \mathcal{CR}(3^n)\}$. Evidently, $\min(h, 3^n) = n$. We found the exact values of $\max(h, 3^n)$ in this section in all situations, namely:

For an even integer $n \geq 4$ and $n \neq 14$. There exists an integer b such that $h(3^n) = \{k \in \mathbb{Z} : n \leq k \leq b\}$. Moreover, an explicit formula for the integer b is given by the following.

1. $b = n$ if and only if $n = 4, 6, 8$.

2. $b = n + 2$ if and only if $n = 10, 12$.

3. If $n = 14 + 2i$ and $i \geq 1$, then $b = 18 + 3i$.

Thus the problem of determining the range of Hamiltonian numbers in the class of connected cubic graphs of order n has been solved. The cubic graphs G of order n which attain the maximum value $h(G)$ are those graphs containing as many cut edges as possible. We continued our investigation by narrowing the class of connected cubic graphs to the class of 2-connected graphs. We produced three classes of 2-connected cubic graphs with relatively small circumference and proved that they are far from being Hamiltonian.

Let $P = P(5, 2)$ be the Petersen graph of order 10. It is well known (for example see [14]) that P has the following properties.

1. It is a vertex transitive and edge transitive graph.

2. It is not Hamiltonian and $h(P) = 11$.

3. Let $u, v \in V(P)$ and $u \neq v$. Let $W : u = u_1, u_2, \ldots, u_t = v$ be a Hamiltonian walk in P. Then $t = 10$ if $uv \notin E(P)$, otherwise $t = 11$.

Let $P(k)$ be a cubic graph with $10k$ vertices formed by k pairwise disjoint copies of $P(5,2)-e$ by adding k edges to link them in a ring as shown in Fig.1. More precisely let $P_1, P_2, \ldots, P_k$ be k pairwise vertex disjoint copies of $P = P(5,2)$. Let $u_i, v_i \in V(P_i)$ and $u_iv_i \in E(P_i)$. Let $P(k)$ be the graph with $V(P(k)) = \bigcup_{j=1}^{k} V(P_j)$ and $E(P(k)) = (\bigcup_{j=1}^{k} E(P_j - u_jv_j)) + \{u_1v_2, u_2v_3, \ldots, u_{k-1}v_k, u_kv_1\}$ (see Figure 23). Then $P(k)$ is a 2-connected cubic graph of order $10k$. We were able to prove that $h(P(k)) = 11k$.

We introduced another class of 2-connected cubic graphs, $H(k)$, order $6k-4$ having circumference $2k+4$ (see Figure 24).

We found $h(H(k))$ for all k as follows: Let k be an integer with $k \geq 2$. Then

$$h(H(k)) = \begin{cases} 7k-5 & if \text{ k } is\ odd, \\ 7k-6 & if \text{ k } is\ even. \end{cases}$$

Thus the circumference is about one third of its order. We studied another class of 2-connected cubic graphs, $G(k)$, $k \geq 2$ (see Figure 25). It was shown in [22] that $G(k)$ is of order $12k-4$ with circumference $2k+8$ which is about one sixth of the order. We found that $h(G(k)) = 14k-4$ for all k.

Based on the results on these two familier of graphs, it is convincing that a connected graph G with small circumference may have a large Hamiltonian number. On the other hand, a connected graph having a higher connectivity may have a small Hamiltonian number. It was proved by Chavátal and Erdős [9] that a graph G with at least three vertices and $\kappa(G) \geq \beta(G)$ is Hamiltonian. This result is the motivation why we considered the problem of determining the range of Hamiltonian numbers in the class of graphs G of order n and $\kappa(G) = k$. Both interpolation and extremal results were obtained in all situations. Let $\mathcal{G}(n, \kappa = k) = \{G \in \mathcal{G}(n) : \kappa(G) = k\}$ and $h(n,k) = \{h(G) : G \in \mathcal{G}(n, \kappa = k)\}$. Furthermore, we denote by $\min(h; n, k) := \min\{h(G) : G \in \mathcal{G}(n,k)\}$ and $\max(h; n, k) := \max\{h(G) : G \in \mathcal{G}(n, \kappa = k)\}$. We proved in that for any pair of integers n, k with $1 \leq k \leq n-1$, there exist positive integers $a := \min(h; n, k)$ and $b := \max(h; n, k)$ such that $h(n, k) = \{x \in \mathbb{Z} : a \leq x \leq b\}$. Moreover, the values of $\min(h; n, k)$ and $\max(h; n, k)$ are obtained in all situations as given belows:

1. Let n be a positive integer with $n \geq 3$. Then $h(n, 1) = \{x \in \mathbb{Z} : n+1 \leq x \leq 2n-2\}$.

2. Let n and k be positive integers. Then $h(n, k) = \{n\}$ if and only if $n \leq 2k$.

3. Let n and k be integers such that $k \geq 2$ and $n > 2k$. Then $\min(h; n, k) = n$ and $\max(h; n, k) = 2(n-k)$. Moreover, for any positive integer i such that $0 \leq i \leq n-2k$, there exists $G_i \in \mathcal{G}(n, \kappa = k)$ with $h(G_i) = 2(n-k) - i$.

4. Let $n \geq 3$ and $k \geq 2$ be integers with $n > 2k$. If $G \in \mathcal{G}(n, \kappa = k)$ and $h(G) = 2(n-k)$, then $m(G) \leq m(CS(n-k, k))$. Further, if $G \in \mathcal{G}(n, \kappa = k)$, then $h(G) = 2(n-k)$ and $m(G) = m(CS(n-k, k))$ if and only if $G \cong CS(n-k, k)$.

4.2 Open Problems

We have encountered some open interesting problems during our study on the graph parameter h for future study. It has been proved by Goodman and Hedetniemi [2] that for a connected graph G of order n, $h(G) \leq 2n-2$. Furthermore, $h(G) = 2n-2$ if and only if G is a tree of order n. Let G be a connected graph of order n. We have defined the Hamiltonian coefficient of G, denoted by $hc(G)$, as $hc(G) = \frac{h(G)}{n}$. For every graph G of order n, $hc(G) \leq \frac{2n-2}{n} < 2$ and $hc(G) = \frac{2n-2}{n}$ if and only if G is a tree. Furthermore, if T_n be a tree of order n, then

$$\lim_{n \to \infty} hc(T_n) = 2.$$

Let G_i be a connected cubic graph of order $14+2i$ with $h(G_i) = 18+3i$. In other words, G_i is a connected cubic graph of order $14+2i$ with $h(G_i) = \max\{h(G) : G \in \mathcal{CR}(3^{14+2i})\}$. Thus $hc(G_i) = \frac{h(G_i)}{|n(G_i)|} = \frac{18+3i}{14+2i} < \frac{3}{2}$ and

$$\lim_{i\to\infty} hc(G_i) = \frac{3}{2}.$$

The problem of finding the maximum value of $hc(G)$ in the class of 2-connected cubic graphs of order n is not easy. By the results we have presented in the last section on the Hamiltonian numbers of three classes of 2-connected cubic graphs, we have that

1. $hc(P(k)) = \frac{11}{10} = 1.1$,

2. $hc(H(k)) = \frac{7k-6}{6k-4} < \frac{7}{6}$ for even integers $k \geq 2$, $hc(H(k)) = \frac{7k-5}{6k-4} < \frac{7}{6}$ for odd integers $k \geq 3$, and
$$\lim_{k\to\infty} hc(H(k)) = \frac{7}{6}.$$

3. $hc(G(k)) = \frac{14k-4}{12k-4} > \frac{7}{6}$ for integers $k \geq 2$, and
$$\lim_{k\to\infty} hc(G(k)) = \frac{7}{6}.$$

A natural next step of our work is the following problem :

Open Problem 1. Let $\mathcal{CR}_3(3^n)$ be the class of 3-connected cubic graphs of order n. Determine $h(\mathcal{CR}_3(3^n))$.

Based on our results, we give the following conjecture :

Open Problem 2. Let $\mathcal{CR}_2(3^n)$ be the class of 2-connected cubic graphs of order n and $G_n \in \mathcal{CR}_2(3^n)$ with $h(G_n) = \max\{h(G) : G \in \mathcal{CR}_2(3^n)\}$. Prove that
$$\lim_{n\to\infty} hc(G_n) = \frac{7}{6}.$$

Let $\mathcal{G}(n, \kappa = k)$ be the class of graphs G of order n and $\kappa(G) = k$. It was shown that if $k \geq 2$, then $\mathcal{G}(n, \kappa = k) = \{n\}$ if and only if $n \leq 2k$. Furthermore, it has been also shown that the complete split graph $CS(n-k, k) \in \mathcal{G}(n, \kappa = k)$ satisfies the following properties.

1. $CS(n-k, k)$ is a k-connected graph of order n with $h(CS(n-k, k)) = 2(n-k)$,

2. If G is a k-connected graph of order n, then $h(G) \leq h(CS(n-k,k))$,

3. If G is a k-connected graph of order n and $h(G) = h(CS(n-k,k))$, then $m(G) \leq m(CS(n-k,k))$.

4. If $k \geq 2$, then $h(\mathcal{G}(n, \kappa = k) = \{n, n+1, \ldots, 2(n-k)\}$.

These properties led us to the following open problem :

Open Problem 3. Let $G \in \mathcal{G}(n, \kappa = k)$ with $h(G) = n + i$. What is the maximum number of edges that the graph G can have?

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VITAE

VITAE

Name : Sermsri Thaithae

Date of Birth : May 12, 1973

Place of Birth : Bangkok

Address : 38/22 Talingchan, Bangkok 10170

Educational Background :

1995 B.Sc(Mathematics), Srinakharinwirot University

1998 M.S.(Mathematics), Chulalongkorn University